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Stability diagram for 4D linear periodic systems with applications to homographic solutions

Regina Martínez [a,*], Anna Samà [a], Carles Simó [b]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, Barcelona, Spain
[b] Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain

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Abstract

We consider a family of 4-dimensional Hamiltonian time-periodic linear systems depending on three parameters, λ_1, λ_2 and ε such that for $\varepsilon = 0$ the system becomes autonomous. Using normal form techniques we study stability and bifurcations for $\varepsilon > 0$ small enough. We pay special attention to the d'Alembert case. The results are applied to the study of the linear stability of homographic solutions of the planar three-body problem, for some homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$, including the Newtonian case.

1. Introduction

Let us consider a family of real periodic linear systems

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \quad A(t) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(t) & -2J_2 \end{pmatrix}, \; \tilde{A}(t) = \begin{pmatrix} \lambda_1 G_1(t,\varepsilon) & 0 \\ 0 & \lambda_2 G_2(t,\varepsilon) \end{pmatrix}, \tag{1}$$

where $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, λ_1, λ_2 are real parameters different from zero, ε is a small positive parameter, and

$$G_i(t,\varepsilon) = 1 - \sum_{j\in\mathbb{N}} \varepsilon^j c_{i,j}(t), \quad i = 1, 2, \tag{2}$$

G_1, G_2 being analytic in (t,ε) with $c_{i,j}(t)$ even periodic functions of t with period T.

* Corresponding author.
E-mail address: reginamb@mat.uab.es (R. Martínez).

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If $\varepsilon = 0$ then system (1) is linear with constant coefficients and one can obtain easily the stability and instability regions in the (λ_1, λ_2)-plane. These regions are described in Section 1.1. Our purpose is to study the bifurcations for ε small and positive.

System (1) can be written as a linear Hamiltonian system with Hamiltonian function

$$H(\mathbf{y}, t) = \frac{1}{2}\left(y_3^2 + y_4^2\right) + y_1 y_4 - y_2 y_3 - V(y_1, y_2, t, \varepsilon), \tag{3}$$

where

$$V(y_1, y_2, t, \varepsilon) = \left[\lambda_1 G_1(t, \varepsilon) - 1\right]\frac{y_1^2}{2} + \left[\lambda_2 G_2(t, \varepsilon) - 1\right]\frac{y_2^2}{2}. \tag{4}$$

The analysis of system (1) has several applications. One of them is the study of the stability of equilibria of mechanical systems defined by a Hamiltonian function of type (3) with a potential $\mathcal{V}(y_1, y_2, t, \varepsilon)$ even in t and such that the quadratic part in y_1 and y_2 has the form (4). In this case, the linearized system at the equilibrium point can be written as (1).

On the other hand, (1) can be obtained as first order variational equations along a periodic solution of an autonomous system. As we shall see in Section 7, one example is given by the homographic solutions of the planar three-body problem with homogeneous potential of degree $-\alpha$, with $0 < \alpha < 2$. After some reductions the linear stability of these orbits is given by the study of a nonautonomous linear system of type (1).

We are mainly interested in the d'Alembert case. It typically appears when the expansions in a perturbative theory are carried out. If the linear term is of the form $\varepsilon(\cos(\omega t), \sin(\omega t))$ it will give rise to harmonics of higher order when we progress in the expansions. But terms like $(\cos(k\omega t), \sin(k\omega t))$ only appear multiplied, at least, by ε^k. For an additional motivation on the d'Alembert property or d'Alembert characteristic see, e.g., [3] in the context of celestial mechanics.

Definition 1. We say that G_i, $i = 1, 2$, satisfy d'Alembert property if an harmonic of order k contains at least ε^k as factor.

Remark 1. In general, $c_{i,j}$ in (2) can contain all harmonics. But if (1) comes from the variational equations along periodic orbits emanating from a fixed point, then (as follows from Lindstedt–Poincaré method) G_i satisfy d'Alembert property (see, e.g., [10] about Lyapunov periodic orbits). In this case ε can be seen as a parameter related to the size of the periodic orbit. Hence, the domain of applicability of the results extends to this larger setting.

Let us denote by $\mu_1, \mu_1^{-1}, \mu_2, \mu_2^{-1}$ the characteristic multipliers of the system defined by (3), that is, the eigenvalues of the monodromy matrix, and define $\mathrm{tr}_i = \mu_i + \mu_i^{-1}$, $i = 1, 2$, as the stability parameters. Notice that tr_i, $i = 1, 2$, depend on the parameters λ_1, λ_2 and ε. Moreover, if $\mathrm{tr}_j \in \mathbb{C} \setminus \mathbb{R}$, $j = 1, 2$, then $\mathrm{tr}_2 = \overline{\mathrm{tr}}_1$.

According to the values of the stability parameters, we shall use the following notation for the different regions in the parameter space $(\lambda_1, \lambda_2, \varepsilon)$:

- EE (elliptic–elliptic) if $|\mathrm{tr}_j| < 2$, $j = 1, 2$;
- EH (elliptic–hyperbolic) if $|\mathrm{tr}_1| < 2$, $|\mathrm{tr}_2| > 2$;

- HH (hyperbolic–hyperbolic) if $|\mathrm{tr}_j| > 2$, $j = 1, 2$;
- CS (complex-saddle) for tr_j, $j = 1, 2$ complex, $\mathrm{tr}_2 = \bar{\mathrm{tr}}_1$.

In the case $\varepsilon = 0$ the stability parameters are trivially obtained. When ε moves away from 0 bifurcations can only appear if some $|\mathrm{tr}_1| = 2$ or $\mathrm{tr}_1 = \mathrm{tr}_2$. These conditions define some curves, to be called *resonant curves*, in the (λ_1, λ_2)-plane.

Let $(\lambda_1, \lambda_2) = (a_1, a_2)$ be a point on a resonant curve for $\varepsilon = 0$. Our purpose is to study $\mathrm{tr}_1, \mathrm{tr}_2$ in a neighbourhood of (a_1, a_2) for $\varepsilon > 0$ small enough. To this end, we introduce small parameters $\delta_1, \delta_2 \in \mathbb{R}$ and we shall consider $\lambda_j = a_j + \delta_j$, $j = 1, 2$. We shall apply the normal form techniques (see [2]) in order to detect changes in the stability. The idea is to perform some canonical transformations to cancel the time dependence up to high order in δ_1, δ_2, ε, if this is possible. The analysis of the normal form obtained in this way gives us domains in the parameter space λ_1, λ_2, ε with different linear stability characteristics as well as their boundaries.

Remark 2. The boundaries, in the parameters $(\lambda_1, \lambda_2, \varepsilon)$, of the regions with different stability character are defined by $|\mathrm{tr}_j| = 2$, for some $j = 1, 2$, or $|\mathrm{tr}_1| = |\mathrm{tr}_2|$. Let $\Phi(T)$ be the monodromy matrix of the linear system defined by (3). Using the symplectic character of $\Phi(T)$ the characteristic polynomial is of the form $P(x) = x^4 + \alpha_1 x^3 + \alpha_2 x^2 + \alpha_1 x + 1$ where $\alpha_1 = -(\mathrm{tr}_1 + \mathrm{tr}_2)$ and $\alpha_2 = 2 + \mathrm{tr}_1\,\mathrm{tr}_2$. As α_1 and α_2 are analytic functions of the parameters ε, δ_1, δ_2, these boundaries belong to the zero locus of some analytic functions of the parameters.

Remark 3. If the functions G_i are not even but general, similar tools can be used to study the possible bifurcations. More terms remain in the normal form and the discussion becomes more involved. See [2] for a simplest case in dimension 2.

1.1. The case $\varepsilon = 0$

The case $\varepsilon = 0$ is studied in an elementary way by using the characteristic polynomial $p(x) = x^4 - (\lambda_1 + \lambda_2 - 4)x^2 + \lambda_1\lambda_2$. We distinguish on the (λ_1, λ_2)-plane the following open regions (see Fig. 1):

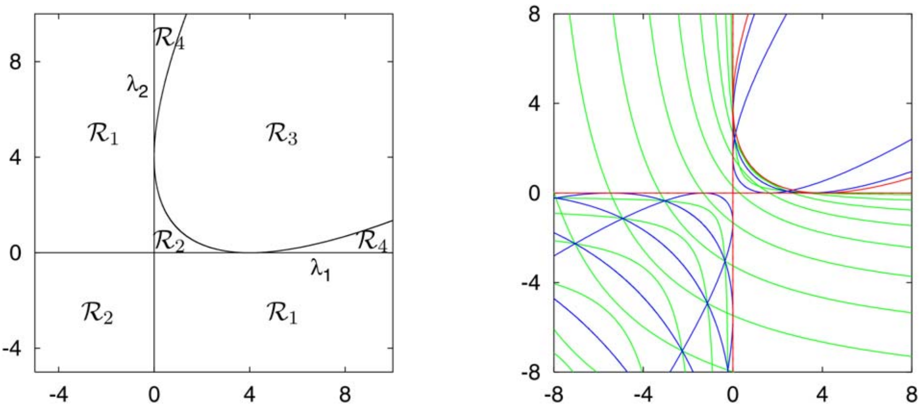

Fig. 1. Stability domains for $\varepsilon = 0$. Some resonant curves for the case of homographic solutions with potential of degree $-\alpha$ being $\alpha = 0.5$ (see Section 7). The period is $T = 2\pi(2-\alpha)^{-1/2}$.

$$\mathcal{R}_1 = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1\lambda_2 < 0\},$$
$$\mathcal{R}_2 = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1\lambda_2 > 0,\ (\lambda_1 + \lambda_2 - 4)^2 > 4\lambda_1\lambda_2,\ \lambda_1 + \lambda_2 - 4 < 0\},$$
$$\mathcal{R}_3 = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1\lambda_2 > 0,\ (\lambda_1 + \lambda_2 - 4)^2 < 4\lambda_1\lambda_2\},$$
$$\mathcal{R}_4 = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1\lambda_2 > 0,\ (\lambda_1 + \lambda_2 - 4)^2 > 4\lambda_1\lambda_2,\ \lambda_1 + \lambda_2 - 4 > 0\}.$$

The following table summarizes the characteristics of these regions

$$\begin{array}{llll}
\mathcal{R}_1: & \mu_1 = e^{\lambda T}, & \mu_2 = e^{i\omega T}, & \mathrm{tr}_1 > 2,\ |\mathrm{tr}_2| \leqslant 2, \\
\mathcal{R}_2: & \mu_1 = e^{i\omega_1 T}, & \mu_2 = e^{i\omega_2 T}, & |\mathrm{tr}_j| \leqslant 2,\ j = 1, 2, \\
\mathcal{R}_3: & \mu_1 = e^{(\alpha+i\beta)T}, & \mu_2 = e^{(\alpha-i\beta)T}, & \mathrm{tr}_j \in \mathbb{C},\ j = 1, 2, \\
\mathcal{R}_4: & \mu_1 = e^{\alpha_1 T}, & \mu_2 = e^{\alpha_2 T}, & \mathrm{tr}_j > 2,\ j = 1, 2
\end{array}$$

where $\lambda, \omega, \omega_1, \omega_2, \alpha, \beta, \alpha_1, \alpha_2 \in \mathbb{R}^+$.

On the λ_1 axis one stability parameter is equal to two, and the other one is $2\cos(\sqrt{4-\lambda_1}T)$ if $\lambda_1 < 4$ and bigger than 2 if $\lambda_1 > 4$. We obtain a symmetric behaviour on the λ_2 axis. If $\lambda_2 = (\sqrt{\lambda_1} - 2)^2$ then $\mathrm{tr}_1 = \mathrm{tr}_2$. In this case, if $0 < \lambda_1 < 4$ then $|\mathrm{tr}_1| = |\mathrm{tr}_2| \leqslant 2$ and $\mathrm{tr}_1 = \mathrm{tr}_2 > 2$ if $\lambda_1 > 4$. On $\lambda_2 = (\sqrt{\lambda_1} + 2)^2$, we obtain $\mathrm{tr}_1 = \mathrm{tr}_2 > 2$ if $\lambda_1 \neq 0$. The points $(4, 0)$, $(0, 4)$ in the (λ_1, λ_2)-plane correspond to degenerate cases in which 1 is a characteristic multiplier with multiplicity 4. Therefore, on these points we have $\mathrm{tr}_1 = \mathrm{tr}_2 = 2$.

Resonant curves in the (λ_1, λ_2)-plane are easily obtained using $|\mathrm{tr}_1| = 2$ or $\mathrm{tr}_1 = \mathrm{tr}_2$.

Let be $\nu = T/\pi$. In $\mathcal{R}_1 \cup \mathcal{R}_2$ we find some resonant curves when $\omega = n/\nu$, for $n \in \mathbb{N}$, and so, one stability parameter equals ± 2. These resonant curves are defined by

$$(\lambda_1 + \omega^2)(\lambda_2 + \omega^2) = 4\omega^2, \quad \omega = n/\nu,\ n \in \mathbb{N}. \tag{5}$$

We note that in $\mathcal{R}_1$ we get a one parameter family of resonant curves (5) with $n \in \mathbb{N}$. However, in $\mathcal{R}_2$, two families are obtained corresponding to ω_1 and ω_2, respectively. For one of them, $n \in \mathbb{N}$. The other family is defined for $n > 2\nu$, $n \in \mathbb{N}$, if $\lambda_1 < 0$, and $n < 2\nu$, $n \in \mathbb{N}$, if $\lambda_1 > 0$. In $\mathcal{R}_2$, bifurcations can also take place for $\omega_1 \pm \omega_2 = 2n/\nu$, $n \in \mathbb{N}$. In this case, $\mathrm{tr}_1 = \mathrm{tr}_2$. This gives a new family of resonant curves defined by

$$\lambda_2 = \lambda_1 + 4(1 - n^2/\nu^2) \pm 4\sqrt{\lambda_1(1 - n^2/\nu^2)}, \tag{6}$$

with $n \leqslant \nu$ if $\lambda_1 > 0$, and $n \geqslant \nu$ if $\lambda_1 < 0$.

In $\mathcal{R}_3$ resonant curves are defined by $\mathrm{tr}_1 = \mathrm{tr}_2$, that is, $T\beta = n\pi$, for $n \in \mathbb{N}$. This happens when $(\lambda_1, \lambda_2) \in \mathcal{R}_3$ satisfies

$$\lambda_2 = \left(\sqrt{\lambda_1} \pm 2\sqrt{1 - \beta^2}\right)^2, \quad \beta = n/\nu, \tag{7}$$

for $n \leqslant \nu$, $n \in \mathbb{N}$. Figure. 1 shows some resonant curves in the different regions.

1.2. Main results

Let us assume that $(\lambda_1, \lambda_2) = (a_1, a_2)$ belongs to a resonant curve for $\varepsilon = 0$. First, the normal form is obtained for the different regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$. The boundaries of the resonant regions up to a given order in the parameters $\delta_1, \delta_2, \varepsilon$ are determined in terms of the coefficients of the normal form.

Then, we restrict to the d'Alembert case (see Remark 1). We note that this is relevant for the homographic solutions. The main results are obtained in this case under some generic assumptions, in the sense that the expected dominant terms are different from zero.

Theorem 1. *Assume the d'Alembert property holds and nondegeneracy conditions are satisfied. Let $(a_1, a_2) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, $a_1 \neq a_2$, be a point corresponding to a single resonance that is, $(a_1, a_2) \in \mathcal{R}_1 \cup \mathcal{R}_2$ with $\omega_1 = n\pi/T$, $n \in \mathbb{N}$, or $(a_1, a_2) \in \mathcal{R}_3$ with $\beta = k\pi/T$, $k \in \mathbb{N}$. Then the width of the resonant regions is at least, of order ε^n and ε^k, respectively.*

The richer case occurs in $\mathcal{R}_2$ at double resonances. For ε small enough an HH region such that $\mathrm{tr}_j > 2$, $j = 1, 2$, is created. The evolution of this region will determine qualitatively the other ones.

Theorem 2. *Assume the d'Alembert property holds and nondegeneracy conditions are satisfied. Let $(a_1, a_2) \in \mathcal{R}_2$, $a_1 \neq a_2$, be a point such that $\omega_j = n_j\pi/T$, $n_j \in \mathbb{N}$, $j = 1, 2$. Then around $(\lambda_1, \lambda_2, \varepsilon) = (a_1, a_2, 0)$ one has:*

(i) *if $n_1 = 3n_2$, regions EE, EH and CS exist and a region HH has either* 0, 1 *or* 2 *connected components*;
(ii) *if $n_1 \neq 3n_2$, then regions EE, EH, CS exist. A region HH always exists except if $n_1 < 3n_2$ and $a_1 > 0$. No local changes in the topology of these domains occur in these cases.*

Remark 4. In (i) of Theorem 2 the number of components of the region HH is determined by some coefficient to be introduced in Section 4.2.

Finally, we study the linear stability of homographic solutions for a planar three-body problem with an homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$. It is well known that there are two kinds of homographic solutions: collinear and triangular. In the Newtonian case, $\alpha = 1$, the small parameter ε is taken as the eccentricity of the Keplerian orbit and $G_1(t; \varepsilon) = G_2(t; \varepsilon) = 1/(1 + \varepsilon \cos t)$, being t the true anomaly. In the non-Newtonian cases ε is taken as a generalized eccentricity.

The methodologies used in this paper and in the companion one [7], allow for an unified study of the linear stability for both kinds of homographic solutions and any $\alpha \in (0, 2)$ (compare with [9], where results are obtained for the triangular Newtonian case).

Moreover, the following result applies to the collinear homographic solutions.

Theorem 3. *Let us consider the Newtonian case and $(a_1, a_2) \in \mathcal{R}_1 \cup \mathcal{R}_2$ such that a single resonant frequency $\omega_1 = n$, $n \in \mathbb{N}$, occurs for $\varepsilon = 0$. Then the two boundaries of resonant regions coincide and there is no bifurcation in this case.*

Some numerical results for the general case, $0 < \alpha < 2$ and finite ε are given. We note that the linear system for homographic solutions has a singularity for $\varepsilon = 1$, $t = \pi$ which corresponds to a collision. Stability properties for these systems when ε is near 1 are studied in [7].

The normal form is obtained in Section 2 and the conditions for bifurcations in Section 3. In Section 4 we consider the d'Alembert case and prove Theorems 1 and 2. Sections 5 and 6 are devoted to the proofs of results of Section 2. In Section 7 we study the linear stability of homographic solutions and we prove Theorem 3.

An announcement of some of the results in this paper can be found in [6].

Concerning physical implications, for the triangular case it turns out that increasing the eccentricity up to moderate values, the domain of linearly stable periodic solutions contains larger masses than in the circular case ($\varepsilon = 0$), but stability is lost for some intermediate values of the masses (see Fig. 5). This holds also for other values of α (see [7, Fig. 6]). In the restricted three-body problem for $\alpha = 1$, this is a well-known property, see [11].

In the collinear case, for which the homographic solutions are of EH type (i.e., partially stable) if $\varepsilon = 0$, there appear narrow tongues of HH type (i.e., totally unstable) in a mass parameter-eccentricity diagram. These tongues are located near the $n + 1/2$ resonances, $n \in \mathbb{N}$, but they do not show up in the n resonances, according to Theorem 3.

2. Normal form

In this section we reduce the Hamiltonian system associated to (3) to normal form. We are interested in using the symmetries of the problem to have simpler formats for the normal form.

We take $\lambda_j = a_j + \delta_j$, $j = 1, 2$, where $(a_1, a_2) \in \mathcal{R}$ is a point on a resonant curve and $|\delta_j|$, $j = 1, 2$, are small enough. The Hamiltonian function (3) can be written as

$$H(\mathbf{y}, t) = H_0(\mathbf{y}) + \tilde{H}(\mathbf{y}, t), \tag{8}$$

where

$$H_0(\mathbf{y}) = \frac{1}{2}\left(y_3^2 + y_4^2\right) + y_1 y_4 - y_2 y_3 + (1 - a_1)\frac{y_1^2}{2} + (1 - a_2)\frac{y_2^2}{2}, \tag{9}$$

$$\tilde{H}(\mathbf{y}, t) = -\frac{\delta_1}{2} y_1^2 - \frac{\delta_2}{2} y_2^2 + (a_1 + \delta_1)\frac{y_1^2}{2} F_1(t; \varepsilon) + (a_2 + \delta_2)\frac{y_2^2}{2} F_2(t; \varepsilon). \tag{10}$$

The Hamiltonian (8) satisfies $H(\mathbf{y}, t) = H(\mathbf{y}, -t)$ and $H(L\mathbf{y}, t) = H(\mathbf{y}, t)$ for all $\mathbf{y} \in \mathbb{R}^4$ and $t \in \mathbb{R}$, where L is the involution with matrix $L = \operatorname{diag}(-1, 1, 1, -1)$.

The first step is to diagonalize $H_0(\mathbf{y})$. Let $\dot{\mathbf{y}} = A_0 \mathbf{y}$ be the linear system defined by H_0. We denote by $\pm\rho_1$, $\pm\rho_2$, the eigenvalues of A_0. In what follows, we will use $\rho_1 = \lambda$, $\rho_2 = i\omega$, $\lambda, \omega \in \mathbb{R}^+$ if $(a_1, a_2) \in \mathcal{R}_1$, $\rho_1 = i\omega_1$, $\rho_2 = i\omega_2$ with $\omega_1, \omega_2 \in \mathbb{R}^+$, $\omega_1 > \omega_2$ if $(a_1, a_2) \in \mathcal{R}_2$, and $\rho_1 = \alpha + i\beta$, $\rho_2 = \bar{\rho}_1$, $\alpha, \beta \in \mathbb{R}^+$, if $(a_1, a_2) \in \mathcal{R}_3$.

Let us denote by $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}^4$ the eigenvectors corresponding to eigenvalues ρ_1, ρ_2, respectively. Then, $\mathbf{v}_1 := L\mathbf{u}_1$ and $\mathbf{v}_2 := L\mathbf{u}_2$ are eigenvectors of eigenvalues $-\rho_1, -\rho_2$, respectively.

The matrix

$$M = (k_1\mathbf{u}_1, k_2\mathbf{u}_2, k_3\mathbf{v}_1, k_4\mathbf{v}_2), \tag{11}$$

where $k_j \in \mathbb{C}$, $j = 1, \ldots, 4$, satisfy $k_1 k_3 \mathbf{u}_1^T J \mathbf{v}_1 = 1$, $k_2 k_4 \mathbf{u}_2^T J \mathbf{v}_2 = 1$ is symplectic. So, we can define a canonical change of variables as $\mathbf{y} = M\mathbf{z}$ which diagonalizes the system associated to H_0, that is, if $\mathcal{H}(\mathbf{z}, t)$ denotes the transformed Hamiltonian, then

$$\mathcal{H}(\mathbf{z}, t) = \mathcal{H}_0(\mathbf{z}) + \tilde{\mathcal{H}}(\mathbf{z}, t), \tag{12}$$

where

$$\mathcal{H}_0(\mathbf{z}) = \rho_1 z_1 z_3 + \rho_2 z_2 z_4, \tag{13}$$

and $\mathbf{z} = (z_1, z_2, z_3, z_4)^T$. We recall that, for definiteness, $\lambda, \omega_1, \omega_2, \alpha, \beta$ are assumed to be positive. This can always be done by defining suitable $\mathbf{z}$ variables.

Lemma 1.

(1) *If* $(a_1, a_2) \in \mathcal{R}_1$ *then* $\mathbf{u}_1^T J \mathbf{v}_1 > 0$ *if* $a_1 > 0$, *and* $\mathbf{u}_1^T J \mathbf{v}_1 < 0$ *if* $a_1 < 0$. *Moreover,* $\mathrm{i}\mathbf{u}_2^T J \mathbf{v}_2 > 0$.
(2) *If* $(a_1, a_2) \in \mathcal{R}_2$ *then* $i\mathbf{u}_1^T J \mathbf{v}_1 > 0$ *and,* $\mathrm{i}\mathbf{u}_2^T J \mathbf{v}_2 > 0$ *if* $a_1 < 0$, *and* $\mathrm{i}\mathbf{u}_2^T J \mathbf{v}_2 < 0$ *if* $a_1 > 0$.

The proof of this lemma is given in Section 6.

After Lemma 1 we can do the following choice for the constants k_j, $j = 1, \ldots, 4$.

(1) If $(a_1, a_2) \in \mathcal{R}_1$, we take $k_1 = (s\mathbf{u}_1^T J \mathbf{v}_1)^{-1/2}$, $k_3 = s k_1$, $k_2 = (\mathrm{i}\mathbf{u}_2^T J \mathbf{v}_2)^{-1/2}$, $k_4 = i k_2$,
(2) if $(a_1, a_2) \in \mathcal{R}_2$, we take $k_1 = (i\mathbf{u}_1^T J \mathbf{v}_1)^{-1/2}$, $k_3 = i k_1$, $k_2 = (si\mathbf{v}_2^T J \mathbf{u}_2)^{-1/2}$, $k_4 = -si k_2$,
(3) if $(a_1, a_2) \in \mathcal{R}_3$, we take $k_1 = (\mathbf{u}_1^T J \mathbf{v}_1)^{-1/2}$, $k_3 = k_1$, $k_2 = (\mathbf{u}_2^T J \mathbf{v}_2)^{-1/2}$, $k_4 = k_2$,

where $s = \mathrm{sgn}(a_1)$. We note that if $(a_1, a_2) \in \mathcal{R}_3$ then $\mathbf{u}_1^T J \mathbf{v}_1$ and $\mathbf{u}_2^T J \mathbf{v}_2$ are complex.

Remark 5. An alternative procedure can also be used. First, one reduces H_0 to a simple form with a real symplectic change. In $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ these forms are, respectively

$$\frac{1}{2}\omega_1\bigl(q_1^2 + p_1^2\bigr) + \lambda q_2 p_2, \qquad \frac{1}{2}\omega_1\bigl(q_1^2 + p_1^2\bigr) + \frac{1}{2}\omega_2\bigl(q_2^2 + p_2^2\bigr) \quad \text{and}$$
$$\alpha(q_1 p_1 + q_2 p_2) + \beta(q_2 p_1 - q_1 p_2).$$

Then the variables are complexified to solve in an easier way the homological equations. The saddle parts do not need any change. The variables in the elliptic parts are changed via

$$q_j = (x_j + i y_j)/\sqrt{2}, \qquad p_j = (i x_j + y_j)/\sqrt{2}.$$

Finally, in the CS case one can use

$$q_1 = (x_1 + i x_2)/\sqrt{2}, \qquad q_2 = (i x_1 + x_2)/\sqrt{2},$$
$$p_1 = (y_1 - i y_2)/\sqrt{2}, \qquad p_2 = (-i y_1 + y_2)/\sqrt{2}.$$

The final Hamiltonians are of the form

$$i\omega_1 x_1 y_1 + \lambda x_2 y_2, \qquad i\omega_1 x_1 y_1 + i\omega_2 x_2 y_2 \quad \text{and} \quad (\alpha + i\beta) x_1 y_1 + (\alpha - i\beta) x_2 y_2,$$

respectively, giving rise to diagonal equations.

From now on, M will be the 4×4 symplectic matrix defined in (11) with k_1, k_2, k_3 and k_4 defined above according to the region considered.

Let us define the following matrices

$$S_1 = M^{-1} L M, \qquad S_2 = -J M^T J \overline{M}. \tag{14}$$

Lemma 2. *The new variable* $\mathbf{z}$ *satisfies* $\bar{\mathbf{z}} = \bar{S}_2 \mathbf{z}$, *and the following equalities hold*

$$\mathcal{H}(\mathbf{z}, t) = \mathcal{H}(S_1 \mathbf{z}, -t), \qquad \mathcal{H}(\mathbf{z}, t) = \overline{\mathcal{H}}(\bar{S}_2 \mathbf{z}, t), \tag{15}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $t \in \mathbb{R}$. *Moreover*

(1) *if* $(a_1, a_2) \in \mathcal{R}_1$ *then* $S_1 \mathbf{z} = (s z_3, i z_4, s z_1, -i z_2)^T$, $\bar{S}_2 \mathbf{z} = (z_1, i z_4, z_3, i z_2)^T$,
(2) *if* $(a_1, a_2) \in \mathcal{R}_2$ *then* $S_1 \mathbf{z} = (i z_3, -i s z_4, -i z_1, i s z_2)^T$, $\bar{S}_2 \mathbf{z} = (i z_3, -i s z_4, i z_1, -i s z_2)^T$,
(3) *if* $(a_1, a_2) \in \mathcal{R}_3$ *then* $S_1 \mathbf{z} = (z_3, z_4, z_1, z_2)^T$, $\bar{S}_2 \mathbf{z} = (z_2, z_1, z_4, z_3)^T$.

The proof of this lemma is given in Section 6.

In order to get the normal form we introduce the variable K as the conjugate variable of time t and we consider the Hamiltonian

$$\mathcal{H}(\mathbf{z}, t, K) = \mathcal{H}_0(\mathbf{z}, K) + \tilde{\mathcal{H}}(\mathbf{z}, t), \tag{16}$$

where $\mathcal{H}_0(\mathbf{z}, K) = \mathcal{H}_0(\mathbf{z}) + K$ and $\mathcal{H}_0(\mathbf{z})$ is given in (13).

Let be $w = e^{2it/\nu}$ (we recall that $\nu = T/\pi$). We can write the Hamiltonian as

$$\mathcal{H}(\mathbf{z}, w, K) = \mathcal{H}_0(\mathbf{z}, K) + \sum_{k=1}^{\infty} \mathcal{H}_k(\mathbf{z}, w), \tag{17}$$

where $\mathcal{H}_k(\mathbf{z}, w)$ contains terms of order k in δ_1, δ_2 and ε. Moreover, $\mathcal{H}_k(\mathbf{z}, w)$ is an homogeneous polynomial of degree 2 in $\mathbf{z}$ whose coefficients depend on w and w^{-1}.

We can use any Lie series method to perform some canonical transformations in order to cancel the time dependence on the Hamiltonian up to high order. This is done in Section 5.

In what follows we shall denote the new variables obtained by the canonical changes of variables involved in the normalization as z_j, $j = 1, \ldots, 4$, again. The next proposition gives the normal form depending on the region $\mathcal{R}_1$, $\mathcal{R}_2$ or $\mathcal{R}_3$. The proof is given in Section 5.

Proposition 1. *Let us denote by NF the normal form up to some arbitrary order in the small parameters* $\delta_1, \delta_2, \varepsilon$.

(1) *If* $(a_1, a_2) \in \mathcal{R}_1$ *and* $\nu\omega \in \mathbb{N}$, *then*

$$NF = K + \lambda z_1 z_3 + i\omega z_2 z_4 + \sigma_1 z_1 z_3 + i\sigma_2 z_2 z_4 + \sigma_3 z_2^2 w^{-\nu\omega} - \sigma_3 z_4^2 w^{\nu\omega}, \tag{18}$$

where $\sigma_j \in \mathbb{R}$, $j = 1, \ldots, 3$, *depend on* δ_1, δ_2 *and* ε.

(2) *If* $(a_1, a_2) \in \mathcal{R}_2$, *then*

$$NF = \begin{cases} N_0 + N_1 & \text{if } \nu\omega_1 \in \mathbb{N},\ \nu\omega_2 \notin \mathbb{N}, \\ N_0 + N_2 & \text{if } \nu\omega_1 \notin \mathbb{N},\ \nu\omega_2 \in \mathbb{N}, \\ N_0 + N_1 + N_2 & \text{if } \nu\omega_1 \in \mathbb{N},\ \nu\omega_2 \in \mathbb{N} \text{ and } \nu\omega_1 \not\equiv \nu\omega_2 \pmod 2, \\ N_0 + N_3 & \text{if } \nu\omega_{hs} \in \mathbb{N},\ \nu\omega_{hd} \notin \mathbb{N}\ (\nu\omega_1 \notin \mathbb{N},\ \nu\omega_2 \notin \mathbb{N}), \\ N_0 + N_4 & \text{if } \nu\omega_{hd} \in \mathbb{N},\ \nu\omega_{hs} \notin \mathbb{N}\ (\nu\omega_1 \notin \mathbb{N},\ \nu\omega_2 \notin \mathbb{N}), \\ N_0 + N_1 + N_2 + N_3 + N_4 & \text{if } \nu\omega_1 \in \mathbb{N},\ \nu\omega_2 \in \mathbb{N} \text{ and } \nu\omega_1 \equiv \nu\omega_2 \pmod 2, \end{cases} \tag{19}$$

where $\omega_{hs} = (\omega_1 + \omega_2)/2$, $\omega_{hd} = (\omega_1 - \omega_2)/2$, *and*

$$\begin{aligned} N_0 &= K + i\omega_1 z_1 z_3 + i\omega_2 z_2 z_4 + i\sigma_1 z_1 z_3 + i\sigma_2 z_2 z_4, \\ N_1 &= \sigma_3 z_1^2 w^{-\nu\omega_1} - \sigma_3 z_3^2 w^{\nu\omega_1}, \\ N_2 &= \sigma_4 z_2^2 w^{-\nu\omega_2} - \sigma_4 z_4^2 w^{\nu\omega_2}, \\ N_3 &= \sigma_5 z_1 z_2 w^{-\nu\omega_{hs}} + s\sigma_5 z_3 z_4 w^{\nu\omega_{hs}}, \\ N_4 &= i\sigma_6 z_1 z_4 w^{-\nu\omega_{hd}} - is\sigma_6 z_2 z_3 w^{\nu\omega_{hd}}, \end{aligned} \tag{20}$$

where $\sigma_j \in \mathbb{R}$, $j = 1, \dots, 6$, *depend on* $\delta_1, \delta_2, \varepsilon$, *and* $s = \operatorname{sgn}(a_1)$.

(3) *If* $(a_1, a_2) \in \mathcal{R}_3$ *and* $\nu\beta \in \mathbb{N}$ *then*

$$\begin{aligned} NF = K &+ (\alpha + i\beta) z_1 z_3 + (\alpha - i\beta) z_2 z_4 + \sigma_1 z_1 z_3 + \bar{\sigma}_1 z_2 z_4 + \sigma_3 z_1 z_4 w^{-\nu\beta} \\ &+ \sigma_3 z_2 z_3 w^{\nu\beta}, \end{aligned} \tag{21}$$

where $\sigma_1 \in \mathbb{C}$, $\sigma_3 \in \mathbb{R}$ *depend on* $\delta_1, \delta_2, \varepsilon$.

Remark 6. Proposition 1 gives the normal form up to a given order, say n, when $\lambda_1 = a_1 + \delta_1$, $\lambda_2 = a_2 + \delta_2$ and (a_1, a_2) is a resonant point for $\varepsilon = 0$. The normal form can be written as $NF = N_0 + \mathcal{N}_n(w)$, where

$$\begin{aligned} N_0 &= K + (\lambda + \sigma_1) z_1 z_3 + i(\omega + \sigma_2) z_2 z_4 && \text{if } (a_1, a_2) \in \mathcal{R}_1, \\ N_0 &= K + i(\omega_1 + \sigma_1) z_1 z_3 + i(\omega_2 + \sigma_2) z_2 z_4 && \text{if } (a_1, a_2) \in \mathcal{R}_2, \\ N_0 &= K + (\alpha + i\beta + \sigma_1) z_1 z_3 + (\alpha - i\beta + \bar{\sigma}_1) z_2 z_4 && \text{if } (a_1, a_2) \in \mathcal{R}_3, \end{aligned}$$

and all the monomials in $\mathcal{N}_n(w)$ depend on w and so, they are time dependent. However, if $\varepsilon = 0$ the initial Hamiltonian (9) is autonomous. In this case, the normal form does not depend on w. Therefore, for the coefficients $\sigma_3, \sigma_4, \sigma_5, \sigma_6$ in Proposition 1 we have

$$\sigma_j = O(\varepsilon^k), \quad j = 3, \dots, 6, \tag{22}$$

for some $k \geqslant 1$ which may depend on the index j. Furthermore, σ_1 and σ_2 depend on $\delta_1, \delta_2, \varepsilon$. In fact σ_1 and σ_2 have terms of order 1 in δ_1, δ_2. These terms can be easily computed by taking into account the variation of the eigenvalues of the system when $\varepsilon = 0$ and we perturb (a_1, a_2) by (δ_1, δ_2). These terms will be explicitly computed in Section 4.

In general, assume that there exists some relation between the order of the different harmonics and the minimal degree in ε of its coefficient in the expansion of the G_i functions, that is, a term in $\cos(n\frac{2\pi}{T}t)$ has $\varepsilon^{m(n)}$ as leading coefficient. Then it is possible to obtain the minimal degree in ε of the σ_j, $j = 3, \dots, 6$, that is, the values of k in (22). To do that one has to examine the paths to reach the relevant resonances in the normal form process. See [1, Appendix B] on how to use these methods in 2D examples in the case of quasi-periodic linear systems.

3. Bifurcations

In order to obtain the boundaries of the different regions in the parameter space when $\varepsilon > 0$ is small enough we shall study the Hamiltonian system associated to the normal form given in Proposition 1. In this section we get the equations for these boundaries.

Before starting this task we have to comment on the effect of the neglected remainder. If the normal form is computed to order n and this is enough to show that the resonant curves split when the effects of $\varepsilon \neq 0$ are taken into account, the effect of the remainder is $O(\varepsilon^{n+1})$. The idea is similar to the study of the branches of analytics curves. If the branches separate at order n, an application of the Implicit Function Theorem, after suitable scaling, shows that higher order terms do not affect the separation between the branches.

Let us take $(a_1, a_2) \in \mathcal{R}_1$ a resonant point for $\varepsilon = 0$. For $\varepsilon > 0$, bifurcation occurs when a pair of characteristic multipliers on the unit circle collide and become real. In this case, the system goes from EH to HH.

Normal form (18) defines the following uncoupled linear system

$$\begin{aligned} \dot{z}_1 &= (\lambda + \sigma_1)z_1, & \dot{z}_2 &= i(\omega + \sigma_2)z_2 - 2\sigma_3 z_4 w^{\nu\omega}, \\ \dot{z}_3 &= -(\lambda + \sigma_1)z_3, & \dot{z}_4 &= -2\sigma_3 z_2 w^{-\nu\omega} - i(\omega + \sigma_2)z_4, \end{aligned} \tag{23}$$

where $\nu\omega = n \in \mathbb{N}$. The system for z_1, z_3 gives real characteristic exponents and, then, a stability parameter is greater than two. This gives an hyperbolic behavior. In order to study the system for z_2, z_4 we perform the change of variables $u = z_2 w^{-\nu\omega/2}$, $v = z_4 w^{\nu\omega/2}$ (the so-called 'co-rotating coordinates'). Then, this system transforms in the following linear system with constant coefficients

$$\begin{aligned} \dot{u} &= i\sigma_2 u - 2\sigma_3 v, \\ \dot{v} &= -2\sigma_3 u - i\sigma_2 v. \end{aligned} \tag{24}$$

Remark 7. The characteristic multipliers μ, μ^{-1} associated to z_2, z_4 are obtained from the monodromy matrix $\Phi_u(T)$ of (24) when $\nu\omega = 2k$, $k \in \mathbb{N}$. If $\nu\omega = 2k + 1$, $k \in \mathbb{N} \cup \{0\}$ then $\Phi_u(T)$ gives $-\mu, -\mu^{-1}$. So, in this case, we shall take from now on the matrix $\Phi_u(2T)$ which has eigenvalues μ^2, μ^{-2}.

For $\varepsilon > 0$ an instability region HH in the parameter space is created. The boundaries of this region up to a given order in $\delta_1, \delta_2, \varepsilon$ are defined by the equation

$$\sigma_2^2 - 4\sigma_3^2 = 0. \tag{25}$$

Table 1
Summary of the cases in which (27) splits in two order 2 systems

$\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \notin \mathbb{N}$	EE $\leftrightarrow$ EH	$\sigma_1^2 - 4\sigma_3^2 = 0$
$\nu\omega_1 \notin \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$	EE $\leftrightarrow$ EH	$\sigma_2^2 - 4\sigma_4^2 = 0$
$\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$ with different parity	EE $\leftrightarrow$ EH	$\sigma_1^2 - 4\sigma_3^2 = 0$ or $\sigma_2^2 - 4\sigma_4^2 = 0$
	EE $\leftrightarrow$ HH	$\sigma_1^2 - 4\sigma_3^2 = 0$ and $\sigma_2^2 - 4\sigma_4^2 = 0$
$\nu\omega_1 \notin \mathbb{N}$, $\nu\omega_2 \notin \mathbb{N}$, $\frac{\nu}{2}(\omega_1 + \omega_2) \in \mathbb{N}$	EE $\leftrightarrow$ CS	$(\sigma_1 + \sigma_2)^2 + 4s\sigma_5^2 = 0$
$\nu\omega_1 \notin \mathbb{N}$, $\nu\omega_2 \notin \mathbb{N}$, $\frac{\nu}{2}(\omega_2 - \omega_2) \in \mathbb{N}$	EE $\leftrightarrow$ CS	$(\sigma_1 - \sigma_2)^2 - 4s\sigma_6^2 = 0$

Now we consider $(a_1, a_2) \in \mathcal{R}_2$ a resonant point for $\varepsilon = 0$. We study the general case in (19), that is, $NF = N_0 + N_1 + N_2 + N_3 + N_4$ where N_i, $i = 0, \ldots, 4$, are given in (20). The other cases in (19) are obtained by taking the suitable coefficients equal to zero. The linear system defined by NF is the following:

$$\begin{aligned}
\dot z_1 &= \mathrm{i}(\omega_1 + \sigma_1)z_1 - \mathrm{i}s\sigma_6 z_2 w^{\frac{\nu}{2}(\omega_1-\omega_2)} - 2\sigma_3 z_3 w^{\nu\omega_1} + s\sigma_5 z_4 w^{\frac{\nu}{2}(\omega_1+\omega_2)},\\
\dot z_2 &= \mathrm{i}\sigma_6 z_1 w^{-\frac{\nu}{2}(\omega_1-\omega_2)} + \mathrm{i}(\omega_2 + \sigma_2)z_2 + s\sigma_5 z_3 w^{\frac{\nu}{2}(\omega_1+\omega_2)} - 2\sigma_4 z_4 w^{\nu\omega_2},\\
\dot z_3 &= -2\sigma_3 z_1 w^{-\nu\omega_1} - \sigma_5 z_2 w^{-\frac{\nu}{2}(\omega_1+\omega_2)} - \mathrm{i}(\omega_1 + \sigma_1)z_3 - \mathrm{i}\sigma_6 z_4 w^{-\frac{\nu}{2}(\omega_1-\omega_2)},\\
\dot z_4 &= -\sigma_5 z_1 w^{-\frac{\nu}{2}(\omega_1+\omega_2)} - 2\sigma_4 z_2 w^{-\nu\omega_2} + \mathrm{i}s\sigma_6 z_3 w^{\frac{\nu}{2}(\omega_1-\omega_2)} - \mathrm{i}(\omega_2 + \sigma_2)z_4.
\end{aligned} \tag{26}$$

We introduce new ('co-rotating') variables $u_1 = z_1 w^{-\nu\omega_1/2}$, $u_2 = z_2 w^{-\nu\omega_2/2}$, $v_1 = z_3 w^{\nu\omega_1/2}$, $v_2 = z_4 w^{\nu\omega_2/2}$. Then, system (26) becomes the following constant coefficients linear system

$$\begin{aligned}
\dot u_1 &= \mathrm{i}\sigma_1 u_1 - \mathrm{i}s\sigma_6 u_2 - 2\sigma_3 v_1 + s\sigma_5 v_2,\\
\dot u_2 &= \mathrm{i}\sigma_6 u_1 + \mathrm{i}\sigma_2 u_2 + s\sigma_5 v_1 - 2\sigma_4 v_2,\\
\dot v_1 &= -2\sigma_3 u_1 - \sigma_5 u_2 - \mathrm{i}\sigma_1 v_1 - \mathrm{i}\sigma_6 v_2,\\
\dot v_2 &= -\sigma_5 u_1 - 2\sigma_4 u_2 + \mathrm{i}s\sigma_6 v_1 - \mathrm{i}\sigma_2 v_2.
\end{aligned} \tag{27}$$

This system splits in two uncoupled systems of order 2 in all the cases given in (19) except for the last one corresponding to $\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$, and $\nu\omega_1 \equiv \nu\omega_2 \pmod 2$. In the cases that (27) becomes uncoupled it is easy to get the equations for the boundaries of the different regions. They are summarized in Table 1. We remark that if $\omega_{hs} = k/\nu$, $k \in \mathbb{N}$, the corresponding equation has no real solution if $s = 1$, that is, $a_1 > 0$, and so, there is no bifurcation in this case. In a similar way there is no bifurcation for $\omega_{hd} = k/\nu$ if $a_1 < 0$. This fact is well known as a consequence of Krein's theorem (see [5]).

Let us consider the case in which $\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$ with the same parity. We denote by $q(x) = x^4 + d_1 x^2 + d_2$ the characteristic polynomial of (27). A simple computation shows that

$$d_1 = \sigma_1^2 + \sigma_2^2 - 4\left(\sigma_3^2 + \sigma_4^2\right) + 2s\left(\sigma_5^2 - \sigma_6^2\right), \qquad d_2 = D_1 D_2, \tag{28}$$

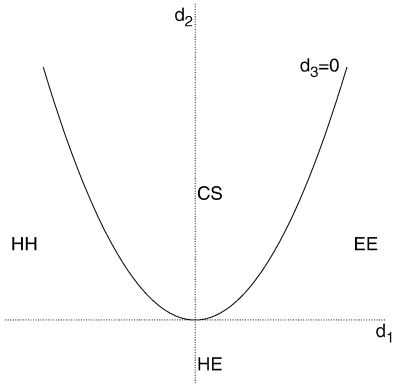

Fig. 2. Stability regions in the (d_1, d_2)-plane.

where

$$D_1 = (\sigma_1 - 2s\sigma_3)(\sigma_2 + 2\sigma_4) + s(\sigma_5 + \sigma_6)^2,$$
$$D_2 = (\sigma_1 + 2s\sigma_3)(\sigma_2 - 2\sigma_4) + s(\sigma_5 - \sigma_6)^2. \tag{29}$$

Let $d_3 = d_1^2 - 4d_2$ be the discriminant of $q(x) = 0$. Then, the different possibilities for the character of the system, excluding boundary values, are represented in Fig. 2.

Finally, we take $(a_1, a_2) \in \mathcal{R}_3$ a resonant point for $\varepsilon = 0$. By performing the change of variables $u_1 = z_1 w^{-\nu\beta/2}$, $u_2 = z_2 w^{\nu\beta/2}$, $v_1 = z_3 w^{\nu\beta/2}$, $v_2 = z_4 w^{-\nu\beta/2}$, to the linear system associated to (21), we obtain an uncoupled linear system with constant coefficients. A transition CS $\leftrightarrow$ HH occurs and the equations for the boundaries of the HH region are given by

$$\mathrm{Im}(\sigma_1) = \pm\sigma_3. \tag{30}$$

4. The d'Alembert case

Now we consider the case when the perturbation, beyond being even in t, satisfies the d'Alembert property (see Remark 1). So, we assume that the functions G_j, $j = 1, 2$, in (2), are of the form

$$\sum_{m\geqslant 0} \varepsilon^m \sum_{l=0}^{m} c_{m,l} \cos\left(l\frac{2\pi t}{T}\right),$$

where $c_{m,l} \in \mathbb{R}$. This property is inherited by the normal form.

After Remark 6 we know that for the coefficients σ_j, $j = 3, 4, 5, 6$, in the normal form, (22) is satisfied for $k \geqslant 1$. The d'Alembert property can be used to determine, under nondegeneracy conditions, the order of these coefficients. In fact, if $\sigma_j w^{\pm n}\mathbf{z}^{\mathbf{l}}$ with $n \in \mathbb{N}$, $\mathbf{l} \in \mathbb{Z}^4$, is a resonant monomial, using the standard notation (see Section 5), then, it is not difficult to see that

$$\sigma_j = c_j \varepsilon^n (1 + O_1), \tag{31}$$

where c_j is a coefficient, depending on the $c_{m,l}$ coefficients and eventually zero, and O_1 contains terms of order 1 in $\delta_1, \delta_2, \varepsilon$. We shall assume in the next, nondegeneracy conditions such that $c_j \neq 0,\ j = 3, 4, 5, 6$.

4.1. Proof of Theorem 1. Single resonances

We shall consider resonant points (a_1, a_2) which belong to a unique resonant curve. This kind of points are found at regions $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$.

We begin with $\mathcal{R}_1$ and assume that (a_1, a_2) belongs to a resonant curve (5), that is,

$$\gamma_n(a_1, a_2) := \big(a_1 + \omega^2\big)\big(a_2 + \omega^2\big) - 4\omega^2 = 0, \quad \text{where } \omega = n/\nu, \tag{32}$$

for some $n \in \mathbb{N}$. From now on, n is fixed.

The boundary surfaces which separate the EH and HH regions for $\varepsilon > 0$ are defined by (25). The coefficient σ_3 is given by (31). The following lemma gives the terms of σ_2 which are of order 1 in δ_1, δ_2.

Lemma 3. *Let $(a_1, a_2) \in \mathcal{R}_1$ be such that $\gamma_n(a_1, a_2) = 0$. Then, the dominant terms in the contribution of δ_1 and δ_2 to σ_2 are*

$$-\left[\frac{\omega^2 + a_2}{D(\omega)}\delta_1 + \frac{\omega^2 + a_1}{D(\omega)}\delta_2\right], \tag{33}$$

where $D(\omega) = 2\omega[2\omega^2 + a_1 + a_2 - 4] \neq 0$.

Remark 8. This lemma is also true if $G_j,\ j = 1, 2$, do not satisfy the d'Alembert property.

Proof. After Remark 6, we consider $\sigma_i = \sigma_i(\delta_1, \delta_2),\ i = 1, 2$, for $\varepsilon = 0$. Then (33) is obtained by looking at the zeroes of the characteristic polynomial $p(x)$ for $\lambda_1 = a_1 + \delta_1,\ \lambda_2 = a_2 + \delta_2$, as perturbations of $\pm\lambda$ and $\pm i\omega$ given by $\lambda + \sigma_1(\delta_1, \delta_2)$ and $i(\omega + \sigma_2(\delta_1, \delta_2))$, respectively. □

In order to describe the boundary surfaces we shall consider perturbations of (a_1, a_2) in an orthogonal direction to the resonant curve (32), that is, $\lambda_1 = a_1 + \delta_1,\ \lambda_2 = a_2 + \delta_2$ with

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \delta \nabla \gamma_n(a_1, a_2), \tag{34}$$

for some parameter δ, being $|\delta|$ small enough. Moreover, (33) becomes $c\|\nabla\gamma_n(a_1, a_2)\|^2\delta$, where $c = -1/D(\omega)$, $\|\nabla\gamma_n(a_1, a_2)\| \neq 0$ and so, we can write

$$\sigma_2 = c\big\|\nabla\gamma_n(a_1, a_2)\big\|^2\delta + \phi_0(\varepsilon) + \delta\phi_1(\varepsilon) + \delta^2 f(\varepsilon, \delta), \tag{35}$$

where ϕ_0 and ϕ_1 are functions of order 1 in ε and $f(\varepsilon, \delta)$ is of order 1 in ε, δ. Here the Euclidean norm is used. The Implicit Function Theorem implies the existence of two analytic functions $\delta_+(\varepsilon), \delta_-(\varepsilon)$, for $\varepsilon \gtrsim 0$ such that

$$\sigma_2\big(\delta_+(\varepsilon),\varepsilon\big)-2\sigma_3\big(\delta_+(\varepsilon),\varepsilon\big)=0,$$
$$\sigma_2\big(\delta_-(\varepsilon),\varepsilon\big)+2\sigma_3\big(\delta_-(\varepsilon),\varepsilon\big)=0. \tag{36}$$

Therefore, in the direction of $\nabla\gamma_n(a_1,a_2)$, the boundaries of the HH region are given by

$$\lambda_1=a_1+\delta_+(\varepsilon), \qquad \lambda_2=a_2+\delta_-(\varepsilon),$$

for $\varepsilon>0$ small enough. Using (31) for $j=3$, (35) and (36) we get the following proposition.

Proposition 2. *Let be $(a_1,a_2)\in\mathcal{R}_1$ such that $\gamma_n(a_1,a_2)=0$ for some $n\in\mathbb{N}$. Assume that the d'Alembert property is satisfied. If c_3 as defined in* (31)*, is nonzero then the width $\delta_+(\varepsilon)-\delta_-(\varepsilon)$ of the HH region is of order ε^n, being the dominant term*

$$-\frac{8c_3\omega(2\omega^2+a_1+a_2-4)}{\|\nabla\gamma_n(a_1,a_2)\|^2}\varepsilon^n.$$

A similar analysis can be done in regions $\mathcal{R}_2$ and $\mathcal{R}_3$ in the case of a single resonance, that is, (a_1,a_2) belongs to a unique resonant curve (5), (6) or (7). In any case we can take (δ_1,δ_2) as (34) for the corresponding resonant curve.

4.2. Proof of Theorem 2. Double resonances

Let $(a_1,a_2)\in\mathcal{R}_2$ be a resonant point which belongs to two or more resonant curves, that is, we assume that

$$\nu\omega_j=n_j, \quad j=1,2, \tag{37}$$

for some $n_1>n_2$ natural numbers. We shall consider the richest case, that is, $n_1\equiv n_2 \pmod 2$. The normal form is $N_0+N_1+N_2+N_3+N_4$ in (19). The analysis of the bifurcations amounts to study the composition of the maps

$$\mathcal{N}:(\lambda_1,\lambda_2,\varepsilon)\mapsto(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6),$$

and

$$\mathcal{P}:(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6)\mapsto(d_1,d_2),$$

where $\mathcal{N}$ denotes the normalization map and $\mathcal{P}$ the characteristic polynomial of the Floquet matrix.

Lemma 4. *Let be $(a_1,a_2)\in\mathcal{R}_2$ and $\omega_1>\omega_2$ the frequencies obtained for $\varepsilon=0$. Then, the dominant terms in the contribution of δ_1 and δ_2 to σ_1,σ_2 are*

$$\mathcal{J}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}, \quad \textit{where } \mathcal{J}=\begin{pmatrix}-(\omega_1^2+a_2)/D_1 & -(\omega_1^2+a_1)/D_1\\ -(\omega_2^2+a_2)/D_2 & -(\omega_2^2+a_1)/D_2\end{pmatrix},$$

$D_1 = 2\omega_1[(a_1+a_2-4)+2\omega_1^2] \neq 0$, $D_2 = 2\omega_2[(a_1+a_2-4)+2\omega_2^2] \neq 0$. *Moreover, the matrix* $\mathcal{J}$ *is regular if* $a_1 \neq a_2$.

The proof follows the same idea as the one of Lemma 3.

After Lemma 4 we can use σ_1 and σ_2 as parameters instead of δ_1, δ_2. Then bifurcations will be described in terms of σ_1 and σ_2. As the functions G_j in (2) satisfy d'Alembert property, we have

$$\sigma_3 = m_1\varepsilon^{n_1}(1+O_1), \qquad \sigma_4 = m_2\varepsilon^{n_2}(1+O_1),$$
$$\sigma_5 = m_3\varepsilon^{(n_1+n_2)/2}(1+O_1), \qquad \sigma_6 = m_4\varepsilon^{(n_1-n_2)/2}(1+O_1),$$

where m_j, $j=1,\dots,4$, are real values and O_1 denote terms of first order in $\varepsilon, \delta_1, \delta_2$. We shall assume nondegeneracy conditions in the sense that $m_j \neq 0$, $j=1,\dots,4$.

First of all we study the magnitude of σ_j, $j=3,\dots,6$. We distinguish different cases.

(1) If $n_1 > 3n_2$, then $n_1 > (n_1+n_2)/2 > (n_1-n_2)/2 > n_2$ and therefore $|\sigma_3| \ll |\sigma_5| \ll |\sigma_6| \ll |\sigma_4|$.
(2) If $n_1 = 3n_2$, then $n_1 > (n_1+n_2)/2 > (n_1-n_2)/2 = n_2$ and therefore $|\sigma_3| \ll |\sigma_5| \ll |\sigma_4|$ and σ_6 is of the same order of magnitude of σ_4.
(3) If $n_1 < 3n_2$, then $n_1 > (n_1+n_2)/2 > n_2 > (n_1-n_2)/2$ and therefore $|\sigma_3| \ll |\sigma_5| \ll |\sigma_4| \ll |\sigma_6|$.

We introduce the following scaled parameters

$$\tilde{\sigma}_j = \frac{\sigma_j}{\sigma_4}, \quad j=1,2,3,5, \qquad A = \frac{\sigma_6}{\sigma_4}, \tag{38}$$

and we define $\mu := \varepsilon^{\frac{n_1-n_2}{2}}$.

We begin with the second case. Then $\mu = \varepsilon^{n_2}$ and hence

$$\tilde{\sigma}_3 = O(\mu^2), \qquad \tilde{\sigma}_5 = O(\mu), \qquad A = O(1).$$

Using the scalings we introduce new functions (see Section 3)

$$\tilde{d}_1 = \frac{d_1}{\sigma_4^2}, \qquad \tilde{D}_1 = \frac{D_1}{\sigma_4^2}, \qquad \tilde{D}_2 = \frac{D_2}{\sigma_4^2}, \qquad \tilde{d}_2 = \tilde{D}_1\tilde{D}_2, \qquad \tilde{d}_3 = \tilde{d}_1^2 - 4\tilde{d}_2.$$

Let be $B := sA^2$ where s is the $\mathrm{sign}(a_1)$ as defined in Section 2. Notice that $B \neq 0$. We can write these functions in terms of μ like

$$\begin{gathered}
\tilde{d}_1 = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - (4+2B) + O(\mu^2),\\
\tilde{D}_1 = \tilde{\sigma}_1(\tilde{\sigma}_2+2) + B + O(\mu),\\
\tilde{D}_2 = \tilde{\sigma}_1(\tilde{\sigma}_2-2) + B + O(\mu), \qquad \tilde{d}_2 = \tilde{D}_1\tilde{D}_2,\\
\tilde{d}_3 = (\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2 + 4)^2 - 4B[(\tilde{\sigma}_1+\tilde{\sigma}_2)^2 - 4] + O(\mu).
\end{gathered} \tag{39}$$

The idea is to study the bifurcation diagram in the $(\tilde\sigma_1, \tilde\sigma_2)$-plane in terms of B.
First we will assume that $\mu = 0$. We obtain the following result.

Proposition 3. *Assume that the hypothesis of Theorem* 2 *are satisfied and* $n_1 = 3n_2$. *Under the generic assumptions* $m_2 \neq 0$, $m_4 \neq 0$ *in the normal form and neglecting* σ_3, σ_5 *terms* (*i.e., setting* $\mu = 0$) *the unique changes in the bifurcation diagram are produced at* $B = -1$ *and* $B = -27/16$.

Figure 3 shows the bifurcation diagram for $\mu = 0$ in different cases. We note that, in particular, no HH regions exists if $B < -1$.

Proof. The different stability regions are determined by the intersections of the zero sets of the functions given in (39) for $\mu = 0$ according to Fig. 2. Notice that we assume $B \neq 0$.

We consider first the set of zeroes of $\tilde d_2$. The hyperbolas $\tilde\sigma_2 = \mp 2 - B/\tilde\sigma_1$ defined by $\tilde D_1 = 0$ and $\tilde D_2 = 0$ respectively have no self intersections. The region $\tilde d_2 < 0$, which corresponds to an EH region, has 2 connected components.

Now we consider the curve $\tilde d_3 = 0$. We note that the zero set of $\tilde d_3$ is symmetric with respect to the origin. Self intersections are determined by the additional conditions $\partial \tilde d_3/\partial\tilde\sigma_1 = 0$ and $\partial \tilde d_3/\partial\tilde\sigma_2 = 0$. These equations only have common solutions for $B = -1$.

Now we go to study the intersections of $\tilde d_2 = 0$ and $\tilde d_3 = 0$. This is equivalent to look for the intersections of $\tilde d_1 = 0$ and $\tilde d_2 = 0$. We recall that $\tilde d_2 = \tilde D_1 \tilde D_2$. So, we shall consider the intersections of

$$\tilde d_1 = 0, \qquad \tilde D_1 = 0. \tag{40}$$

Using the symmetry, the solutions of $\tilde d_1 = 0$, $\tilde D_2 = 0$ will be easily obtained.

The solutions of (40) are the intersection points of a circle of radius $4 + 2B$ and the hyperbola $\tilde\sigma_2 = -2 - \frac{B}{\tilde\sigma_1}$. We assume $B > -2$, otherwise (40) has no real solutions. We begin by looking for the points in $\tilde D_1 = 0$ such that the distance to the origin is a relative minimum. To this end, we use a Lagrange multiplier ρ with Lagrangian

$$L = \tilde\sigma_1^2 + \tilde\sigma_2^2 - \rho \tilde D_1.$$

We get a minimum $(\tilde\sigma_{1,m}, \tilde\sigma_{2,m})$ for

$$\tilde\sigma_{1,m} = \frac{4\rho}{4-\rho^2}, \qquad \tilde\sigma_{2,m} = \frac{2\rho^2}{4-\rho^2}, \tag{41}$$

where ρ satisfies

$$\frac{(4-\rho^2)^2}{\rho} = -\frac{32}{B}. \tag{42}$$

For any value of B, $B \neq 0$, (42) has two real solutions ρ_1, ρ_2 giving rise to points P_1, P_2, respectively, in the $(\tilde\sigma_1, \tilde\sigma_2)$-plane. If $B > 0$ then $\rho_1 < -2 < \rho_2 < 0$ and, $0 < \rho_1 < 2 < \rho_2$ if $B < 0$.

Now we study the sign of $\tilde d_1$ on P_1, P_2. Using (41) for $B \neq 0$ we get

$$\tilde d_1(\rho) := \tilde d_1(\tilde\sigma_{1,m}, \tilde\sigma_{2,m}) = -\frac{B}{8}\bigl[\rho\bigl(\rho^2+4\bigr) + 16\bigr] - 4.$$

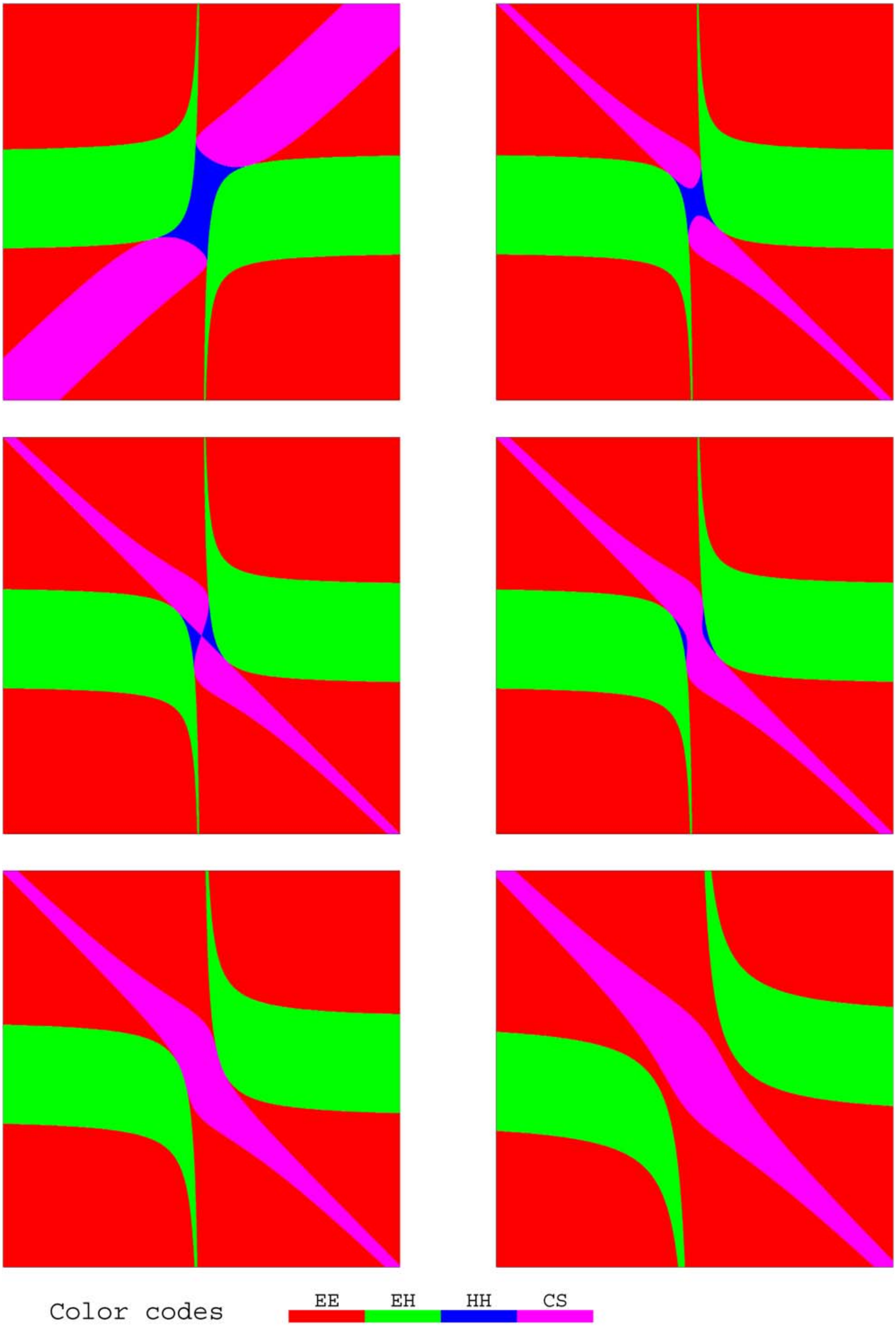

Fig. 3. A sample of the bifurcation diagrams near double resonance in the d'Alembert case with $n_1 = 3n_2$ and $\mu = 0$. Values of B from left to right and top to bottom: $1, -0.9, -1, -1.1, -27/16, -4$. The horizontal (respectively vertical) variable is $\tilde{\sigma}_1$ (respectively $\tilde{\sigma}_2$).

Let ρ_3 be the unique solution of $\tilde{d}_1(\rho) = 0$. If $B > 0$, one has $\rho_1 < \rho_3 < -2 < \rho_2 < 0$. Then, $\tilde{d}_1(\rho_1) > 0$ and $\tilde{d}_1(\rho_2) < 0$, that is, only P_2 is inside the circle defined by $\tilde{d}_1 = 0$. In a similar way, by analyzing the relative position of ρ_1, ρ_2 and ρ_3, it is not difficult to show that

- if $B < -27/16$, P_1 and P_2 live outside the circle and so, (40) has no real solution;
- if $-27/16 < B < 0$, only P_1 is inside the circle and (40) has two different solutions;
- if $B = -27/16$, (40) has a unique real solution $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (3/4, 1/4)$.

Therefore, if $B < -\frac{27}{16}$ there is no HH region (see Fig. 3(f)), if $-\frac{27}{16} < B < -1$ there exists an HH region having two connected components (see Fig. 3(d)) and, if $-1 < B$ the HH region has one connected component (see Fig. 3(a), (b)). $\square$

Now we study the case $\mu \neq 0$, that is, we analyze the effect of the neglected terms. We obtain the following result.

Proposition 4. *Assume that hypothesis in Theorem* 2 *are satisfied and* $n_1 = 3n_2$. *Under the generic assumptions* $m_j \neq 0$, $j = 1, \ldots, 4$, *in the normal form* (19) *the only changes in the bifurcation diagram are produced at* $B = -(1 + \tilde{\sigma}_3)^2$ *and at* $B_\pm = -27/16 \pm sA\tilde{\sigma}_5/2 + O(\mu^2)$.

Proof. We know from Proposition 3 that in the case $\mu = 0$, bifurcations are produced at $B = -1$ due to self-intersections of $\tilde{d}_3 = 0$ and, $B = -27/16$ when $\tilde{d}_1 = 0$ and $\tilde{d}_2 = 0$ have tangencies. We recall that in this case no self-intersections of $\tilde{d}_2 = 0$ occur.

Let us consider $\mu \neq 0$ small enough. In this case, using (29) we see that self-intersections of $\tilde{d}_2 = 0$ occur if

$$\tilde{D}_1 = (\tilde{\sigma}_1 - 2s\tilde{\sigma}_3)(\tilde{\sigma}_2 + 2) + s(\tilde{\sigma}_5 + A)^2 = 0, \qquad \tilde{D}_2 = (\tilde{\sigma}_1 + 2s\tilde{\sigma}_3)(\tilde{\sigma}_2 - 2) + s(\tilde{\sigma}_5 - A)^2 = 0.$$

By subtracting these equations and substituting the relation obtained in $\tilde{D}_1 = 0$ it turns out that

$$\tilde{\sigma}_2^2 O(\mu^2) + \tilde{\sigma}_2 O(\mu) + B + O(\mu^2) = 0.$$

Then, self-intersections of $\tilde{d}_2 = 0$ can occur, but outside a local neighbourhood of the origin on the $(\tilde{\sigma}_1, \tilde{\sigma}_2)$-plane. Hence, they should not be considered.

Now we study the self-intersections of $\tilde{d}_3 = 0$. They are produced if $\tilde{d}_3 = 0$, $\frac{\partial \tilde{d}_3}{\partial \tilde{\sigma}_1} = 0$ and $\frac{\partial \tilde{d}_3}{\partial \tilde{\sigma}_2} = 0$. If $\mu = 0$, this system has the solution $(B, \tilde{\sigma}_1, \tilde{\sigma}_2) = (-1, 0, 0)$. The Jacobian with respect to B, $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ at that point is different from zero. Then, the Implicit Function Theorem ensures the preservation of the intersection which will occur for a value of B equal, a priori, to $-1 + O(\mu)$ and with values $\tilde{\sigma}_1, \tilde{\sigma}_2 = O(\mu)$.

An elementary computation shows that the self-intersections of $\tilde{d}_3 = 0$ occurs exactly for $B = -(1 + \tilde{\sigma}_3)^2$ at $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \tilde{\sigma}_5$. Furthermore, for that value of B, the line $\tilde{\sigma}_1 + \tilde{\sigma}_2 = 2\tilde{\sigma}_5$ is one of the components of $\tilde{d}_3 = 0$. We note that this is true even in the non-d'Alembert case (see Fig. 4 and Remark 10).

It remains to study the modification of the tangencies of the zero sets of $\tilde{d}_1 = 0$ and $\tilde{d}_2 = 0$. We note that symmetry is lost for $\mu \neq 0$. So, one has to consider the cases $\tilde{d}_1 = 0$, $\tilde{D}_1 = 0$ and $\tilde{d}_1 = 0$, $\tilde{D}_2 = 0$ separately. Let us consider the first case. We have

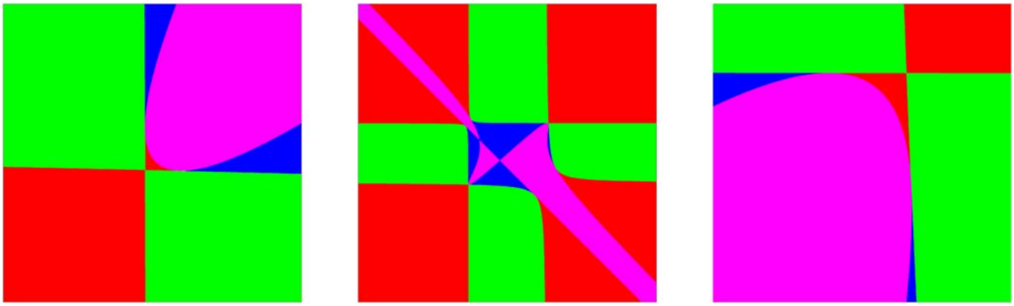

Fig. 4. An example of self-intersections of $\tilde{d}_3 = 0$ in the general case. Scaled parameters used: $\tilde{\sigma}_3 = -1.3$, $\tilde{\sigma}_5 = -0.5$, $A = -0.3$, $s = -1$. Variables plotted and color code as in Fig. 3. The central plot shows a global view, the left and right ones are magnifications. Up to 19 connected components can be seen.

$$\tilde{d}_1 = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - (4 + 2B) + O(\mu^2) = 0,$$
$$\tilde{D}_1 = \tilde{\sigma}_1(\tilde{\sigma}_2 + 2) + B + \nu + O(\mu^2) = 0,$$

where $\nu := 2sA\tilde{\sigma}_5 = O(\mu)$. Up to order μ, $\tilde{d}_1 = 0$ is a circle. Following the same steps as in the proof of Proposition 3, we look for the points of $\tilde{D}_1 = 0$ which are at minimum distance to the origin. Using the Lagrangian $L = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - \rho\tilde{D}_1$ we get a minimum $(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m})$ as (41) where the Lagrange multiplier ρ satisfies

$$\tilde{D}_1(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m}) = \frac{32\rho}{(4-\rho^2)^2} + B + \nu = 0.$$

Furthermore,

$$\tilde{d}_1(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m}) = \frac{16\rho^2}{(4-\rho^2)^2} + \frac{4\rho^4}{(4-\rho^2)^2} - (4 + 2B).$$

We must solve the following system:

$$32\rho + (B + \nu)(4 - \rho^2)^2 = 0, \qquad 16\rho^2 + 4\rho^4 - (4 + 2B)(4 - \rho^2)^2 = 0.$$

For $\mu = 0$, we have the solution $\rho = \frac{2}{3}$, $B = -\frac{27}{16}$. One step of Newton's method around that solution gives the critical value of B

$$B_+ = -\frac{27}{16} + \frac{1}{2}sA\tilde{\sigma}_5 + O(\mu^2).$$

A similar study for $\tilde{d}_1 = 0$, $\tilde{D}_2 = 0$ gives a second critical value

$$B_- = -\frac{27}{16} - \frac{1}{2}sA\tilde{\sigma}_5 + O(\mu^2). \qquad \square$$

Remark 9. The geometrical interpretation is that the two narrow HH domains which in the Fig. 3(f) disappear on the (b) plot ($B = -\frac{27}{16}$) when going from left to right, disappear for slightly different values of B if $\mu \neq 0$. No further changes occur in the bifurcation diagram for ε small enough in case (2).

Proof of Theorem 2. Now the item (i) of Theorem 2 follows from Propositions 3 and 4. To prove (ii) we study the cases $n_1 > 3n_2$ and $n_1 < 3n_2$. To this end we use the same scalings as in case $n_1 = 3n_2$. We have that the parameter A in (38) is of order $O(\varepsilon^{\frac{n_1-3n_2}{2}})$. Then, the case $n_1 > 3n_2$, has the same characteristics than a very small value of $|B|$. In case $n_1 < 3n_2$, we get the same behaviour as the one for a very large value of $|B|$. □

Remark 10. In the non-d'Alembert case the discussion of the different bifurcations follows from the analysis of (28) and (29) without making any assumption on the order of magnitude of the different parameters involved. Assuming $\sigma_4 \neq 0$, scaled parameters can be introduced as in (38). Then the number of self-intersections of $\tilde{d}_3 = 0$ can increase. Figure 4 shows an example.

5. Proof of Proposition 1

Let $\mathcal{H}(\mathbf{z}, w, K)$ be the Hamiltonian defined in (16). Our purpose is to obtain the normal form using the symmetries of $\mathcal{H}(\mathbf{z}, w, K)$. Let be $\mathcal{H}(\mathbf{z}, w) = \mathcal{H}(\mathbf{z}, w, K) - K$. We recall that $\mathcal{H}(\mathbf{z}, w)$ is an homogeneous polynomial of degree 2 in $\mathbf{z}$ whose coefficients depend on w and w^{-1}.

It will be useful to introduce the following functions

$$\mathcal{F}(\mathbf{z}, w) = f_1 z_1^2 + f_2 z_2^2 + f_3 z_3^2 + f_4 z_4^2 + f_5 z_1 z_2 + f_6 z_1 z_3 + f_7 z_1 z_4 + f_8 z_2 z_3 + f_9 z_2 z_4 + f_{10} z_3 z_4, \tag{43}$$

where $f_k = f_k(w)$, $k = 1, \dots, 10$, can be written as

$$f(w) = \sum_{j \geqslant 0} \left(\tilde{c}_j w^j + \tilde{d}_j w^{-j}\right), \tag{44}$$

the coefficients $\tilde{c}_j$, $\tilde{d}_j$ being analytic functions on $\delta_1, \delta_2, \varepsilon$. We shall denote by H_2^T the vector space of functions of the form (43). For a given $\mathcal{F}(\mathbf{z}, w)$ in H_2^T, $\overline{\mathcal{F}}(\mathbf{z}, w)$ will be obtained from (43) by a substitution of f_k by $\overline{f}_k = \overline{f_k(w)}$, for $k = 1, \dots, 10$, where the bar stands for the complex conjugate.

From Lemma 2 and taking into account that w has been defined in Section 2 as $w = e^{\frac{2it}{\nu}}$, we get

$$\mathcal{H}(\mathbf{z}, w) = \mathcal{H}\left(S_1 \mathbf{z}, w^{-1}\right). \tag{45}$$

Moreover, as far as $\mathcal{H}(\mathbf{z}, t)$ in (15) is an even function of t, we get

$$\mathcal{H}(\mathbf{z}, w) = \overline{\mathcal{H}}(\bar{S}_2 \mathbf{z}, w). \tag{46}$$

We shall see that these two symmetries will be preserved to the normal form. This will allow us to compute it in an easy way.

Using, for instance, the Giorgilli–Galgani algorithm [4] we can write $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1 + \mathcal{N}_2 + \cdots + \mathcal{N}_m$, where

$$\mathcal{N}_k = \sum_{j=0}^{k} \mathcal{H}_{j,k-j}, \quad \mathcal{H}_{k,j} = \sum_{l=1}^{j} \frac{l}{j}[G_l, \mathcal{H}_{k,j-l}],\ \mathcal{H}_{k,0} = \mathcal{H}_k, \tag{47}$$

and G_k is a solution of the homological equation

$$M_m + [G_m, \mathcal{H}_0] = R_m$$

being $\mathcal{H}_0 = \mathcal{H}_0(\mathbf{z}, K) = \rho_1 z_1 z_3 + \rho_2 z_2 z_4 + K$,

$$M_m = \sum_{j=0}^{m-1} \mathcal{H}_{m-j,j} + \sum_{l=1}^{m-1} \frac{l}{m}[G_l, \mathcal{H}_{0,m-l}]$$

and where R_m contains resonant terms of order m in δ_1, δ_2 and ε. Note that $\mathcal{H}_{ij}, G_k, M_k$ belong to H_2^T. We denote each term as

$$g = h\mathbf{z}^{\mathbf{l}} w^j, \quad h = c\delta_1^{j_1}\delta_2^{j_2}\varepsilon^{j_3}, \tag{48}$$

where $c \in \mathbb{C}$ is a constant, $j_i \in \mathbb{Z}$, $j_i \geqslant 0$, $i = 1, 2, 3$, $j \in \mathbb{Z}$, and $\mathbf{z}^{\mathbf{l}} = z_1^{l_1} z_2^{l_2} z_3^{l_3} z_4^{l_4}$ with $l_k \in \mathbb{Z}$, $l_k \geqslant 0$, $k = 1, 2, 3, 4$, satisfying $l_1 + l_2 + l_3 + l_4 = 2$. $\mathbf{z}^{\mathbf{l}} w^j$ as in (48) is a resonant monomial if $[\mathbf{z}^{\mathbf{l}} w^j, \mathcal{H}_0] = 0$. From this equation it is easy to get the following lemma.

Lemma 5. *$z_1 z_3$, $z_2 z_4$ are resonant terms for all $(a_1, a_2) \in \mathcal{R}$. Furthermore, additional resonant monomials appear as follows*:

(1) *$z_2^2 w^{-\nu\omega}$, $z_4^2 w^{\nu\omega}$ when $\omega\nu \in \mathbb{N}$ if $(a_1, a_2) \in \mathcal{R}_1$,*
(2) *if $(a_1, a_2) \in \mathcal{R}_2$ then*
  (a) *$z_1^2 w^{-\nu\omega_1}$, $z_3^2 w^{\nu\omega_1}$ when $\omega_1\nu \in \mathbb{N}$,*
  (b) *$z_2^2 w^{-\nu\omega_2}$, $z_4^2 w^{\nu\omega_2}$ when $\omega_2\nu \in \mathbb{N}$,*
  (c) *$z_1 z_2 w^{-\frac{\nu}{2}(\omega_1+\omega_2)}$, $z_3 z_4 w^{\frac{\nu}{2}(\omega_1+\omega_2)}$ when $\frac{\nu}{2}(\omega_1 + \omega_2) \in \mathbb{N}$,*
  (d) *$z_1 z_4 w^{-\frac{\nu}{2}(\omega_1-\omega_2)}$, $z_2 z_3 w^{\frac{\nu}{2}(\omega_1-\omega_2)}$ when $\frac{\nu}{2}(\omega_1 - \omega_2) \in \mathbb{N}$,*
(3) *$z_1 z_4 w^{-\nu\beta}$, $z_2 z_3 w^{\nu\beta}$ when $\nu\beta \in \mathbb{N}$ if $(a_1, a_2) \in \mathcal{R}_3$.*

Let $\mathcal{F}(\mathbf{z}, w)$ be in H_2^T.

Definition 2. $\mathcal{F}(\mathbf{z}, w)$ satisfies the S_2-property if

$$\mathcal{F}(\mathbf{z}, w) = \overline{\mathcal{F}}(\bar{S}_2\mathbf{z}, w), \tag{49}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $w \in \mathbb{C}$, $|w| = 1$.

Definition 3. $\mathcal{F}(\mathbf{z}, w)$ satisfies the S_1^+-property if

$$\mathcal{F}(\mathbf{z}, w) = \mathcal{F}\big(S_1\mathbf{z}, w^{-1}\big), \tag{50}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $w \in \mathbb{C}$, $|w| = 1$.

Definition 4. $\mathcal{F}(\mathbf{z}, w)$ satisfies the S_1^--property if

$$\mathcal{F}(\mathbf{z}, w) = -\mathcal{F}\big(S_1\mathbf{z}, w^{-1}\big), \tag{51}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $w \in \mathbb{C}$, $|w| = 1$.

Lemma 6. *The normal form up to order m, NF, satisfies the S_2-property.*

Proof. From Lemma 2 we have that the initial Hamiltonian satisfies the S_2-property. So, we only need to prove the following statements:

(i) The Poisson bracket on H_2^T preserves the S_2-property.
(ii) Assume that $M \in H_2^T$ satisfies the S_2-property and let G be a solution of the homological equation

$$[G, \mathcal{H}_0] + M = 0. \tag{52}$$

Then, up to resonant terms, G satisfies the S_2-property.

To prove (i) let us consider $\mathcal{F}, \mathcal{G} \in H_2^T$ satisfying the S_2-property. Let be $Q = [\mathcal{G}, \mathcal{F}]$. Using (49) and the fact that $\bar{S}_2 J \bar{S}_2^T = J$ we get

$$Q(\mathbf{z}, w) = \nabla\mathcal{G}(\mathbf{z}, w)^T J \nabla\mathcal{F}(\mathbf{z}, w) = \nabla\overline{\mathcal{G}}(\bar{S}_2\mathbf{z}, w)^T \bar{S}_2 J \bar{S}_2^T \nabla\mathcal{F}(\bar{S}_2\mathbf{z}, w) = \overline{Q}(\bar{S}_2\mathbf{z}, w).$$

Now we prove (ii). Let us define $Y(\mathbf{z}, w) = G(\mathbf{z}, w) - \overline{G}(\bar{S}_2\mathbf{z}, w)$. Then, we only need to prove that $[Y, \mathcal{H}_0] = 0$.

Let be $\mathcal{D} = \mathrm{diag}(\rho_1, \rho_2, -\rho_1, -\rho_2)$ and consider the homological equation for G written as

$$M(\mathbf{z}, w) + \frac{\partial G}{\partial t}(\mathbf{z}, w) + \nabla G(\mathbf{z}, w)^T \mathcal{D}\mathbf{z} = 0. \tag{53}$$

From (53), using the assumption $M(\mathbf{z}, w) = \overline{M}(\bar{S}_2\mathbf{z}, w)$ and $\overline{\mathcal{D}}\bar{S}_2 = \bar{S}_2\mathcal{D}$ we get

$$\frac{\partial G}{\partial t}(\mathbf{z}, w) - \frac{\partial \overline{G}}{\partial t}(\bar{S}_2\mathbf{z}, w) + \big[\nabla G(\mathbf{z}, w)^T - \nabla\overline{G}(\bar{S}_2\mathbf{z}, w)^T \bar{S}_2\big]\mathcal{D}\mathbf{z} = 0.$$

Using the equality above, a simple computation shows that $[Y, \mathcal{H}_0] = 0$ and then $Y(\mathbf{z}, w)$ has only resonant terms. □

Lemma 7. *The normal form NF up to order m satisfies the S_1^+-property.*

Proof. From Lemma 2 the initial Hamiltonian satisfies the S_1^+-property. So, we shall prove the following statements:

(i) If $\mathcal{F} \in H_2^T$ satisfies the S_1^+-property and $\mathcal{G} \in H_2^T$ satisfies the S_1^--property, then $Q := [\mathcal{G}, \mathcal{F}]$ satisfies the S_1^+-property.
(ii) Assume that $M \in H_2^T$ satisfies the S_1^+-property. Let $G \in H_2^T$ be the solution of the homological equation (52). Then, up to resonant terms, G satisfies the S_1^--property.

The proof of (i) and (ii) follows the same steps as the proof of Lemma 6. For (i) one has to use that $S_1 J S_1^T = -J$. To prove (ii) we introduce $Y(\mathbf{z}, w) = G(\mathbf{z}, w) + G(S_1\mathbf{z}, w^{-1})$ and use that $S_1\mathcal{D} = -\mathcal{D}S_1$ to get $[Y, \mathcal{H}_0] = 0$. $\square$

After Lemmas 6 and 7 it is easy to get the relations between the coefficients in the normal form. We give some hint in the case of the region $\mathcal{R}_2$. For the other regions the process is similar.

Let be $(a_1, a_2) \in \mathcal{R}_2$. We consider the case for which the normal form contains all possible resonant terms and we write it as

$$\begin{aligned} NF(\mathbf{z}, w) = {} & K + i\omega_1 z_1 z_3 + i\omega_2 z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 + a_1 z_1^2 w^{-\nu\omega_1} + a_3 z_3^2 w^{\nu\omega_1} \\ & + a_2 z_2^2 w^{-\nu\omega_2} + a_4 z_4^2 w^{\nu\omega_2} + a_5 z_1 z_2 w^{-\nu\omega_{hs}} + a_{10} z_3 z_4 w^{\nu\omega_{hs}} \\ & + a_7 z_1 z_4 w^{-\nu\omega_{hd}} + a_8 z_2 z_3 w^{\nu\omega_{hd}}. \end{aligned}$$

Using the S_1^+-property, that is, $NF(S_1\mathbf{z}, w^{-1}) = NF(\mathbf{z}, w)$ we get

$$a_3 = -a_1, \qquad a_4 = -a_2, \qquad a_{10} = s a_5, \qquad a_8 = -s a_7.$$

In a similar way, using the S_2-property we get

$$a_6 = -\bar{a}_6, \qquad a_9 = -\bar{a}_9, \qquad a_3 = -\bar{a}_1, \qquad a_4 = -\bar{a}_2, \qquad a_{10} = s\bar{a}_5, \qquad a_8 = s\bar{a}_7.$$

Therefore $a_1, a_2, a_5 \in \mathbb{R}$, $a_6, a_7, a_9 \in \mathrm{i}\mathbb{R}$ and the equalities $a_3 = -a_1$, $a_4 = -a_2$, $a_{10} = s a_5$, $a_8 = -s a_7$ hold. This proves (19).

6. Proof of auxiliary lemmas

Proof of Lemma 1. It is easy to check that $\mathbf{u}_\rho = (2\rho, a_1 - \rho^2, a_1 + \rho^2, -\rho(\rho^2 + 2 - a_1))^T$ is an eigenvector of eigenvalue ρ of A_0. Let us define $\mathbf{v}_\rho := L\mathbf{u}_\rho$.

A simple computation shows that

$$\mathbf{u}_\rho^T J \mathbf{v}_\rho = 2\rho q\big(a_1, a_2; \rho^2\big), \tag{54}$$

where $q(a_1, a_2; \rho^2) = -\rho^4 + 2a_1\rho^2 + 4a_1 - a_1^2$.

Using that $p(\rho) = 0$ with $(\lambda_1, \lambda_2) = (a_1, a_2)$ and the fact that

$$\rho^2 = \alpha_\pm, \quad \text{where } \alpha_\pm = \frac{a_1 + a_2 - 4 \pm \sqrt{\Delta}}{2}, \text{ with } \Delta = (a_1 + a_2 - 4)^2 - 4a_1 a_2,$$

we have that

$$q(a_1, a_2; \alpha_\pm) = -\frac{\sqrt{\Delta}}{2}\left[\sqrt{\Delta} \mp (4 + a_1 - a_2)\right],$$

where the sign $-$ stands for α_+ and $+$ for α_-.

If $a_1 > 0$ ($a_1 < 0$) we check that $(4 + a_1 - a_2)^2 > \Delta$ ($(4 + a_1 - a_2)^2 < \Delta$). Therefore, if $a_1 < 0$ then $q(a_1, a_2; \alpha_\pm) < 0$.

Furthermore, if $a_1 > 0$, as far as $(a_1, a_2) \in \mathcal{R}_1 \cup \mathcal{R}_2$, $4 + a_1 - a_2 > 0$. So, $q(a_1, a_2; \alpha_+) > 0$ and $q(a_1, a_2; \alpha_-) < 0$. Now, using (54) the statement of the lemma follows. □

Proof of Lemma 2. The new variables $\mathbf{z} \in \mathbb{C}^4$ are defined by $\mathbf{y} = M\mathbf{z}$ where we recall that $\mathbf{y} \in \mathbb{R}^4$. Then

$$\mathbf{z} = M^{-1}\bar{\mathbf{y}} = -JM^T J\overline{M}\bar{\mathbf{z}} = S_2\bar{\mathbf{z}}, \tag{55}$$

where we have used the symplectic character of M. Now $\bar{\mathbf{z}} = \bar{S}_2\mathbf{z}$ follows from (55).

By the symmetry given by L, we have that

$$\mathcal{H}(S_1\mathbf{z}, t) = H(MS_1\mathbf{z}, t) = H\big(MS_1M^{-1}\mathbf{y}, t\big) = H(L\mathbf{y}, t) = H(\mathbf{y}, t) = \mathcal{H}(\mathbf{z}, t).$$

Then, using the parity of $\mathcal{H}$ we get the first equality in (15).

Furthermore, $\mathcal{H}(\mathbf{z}, t)$ is real. Therefore

$$\mathcal{H}(\mathbf{z}, t) = \overline{\mathcal{H}(\mathbf{z}, t)} = \overline{\mathcal{H}}(\bar{\mathbf{z}}, t) = \overline{\mathcal{H}}(\bar{S}_2\mathbf{z}, t).$$

A simple computation gives

$$S_1 = \begin{pmatrix} 0 & \tilde{S}_1 \\ \tilde{S}_1^{-1} & 0 \end{pmatrix} \quad \text{with } \tilde{S}_1 = \operatorname{diag}\left(\frac{k_3}{k_1}, \frac{k_4}{k_2}\right).$$

This expression gives $S_1\mathbf{z}$ in the different regions.

We note that if $(a_1, a_2) \in \mathcal{R}_1$ then $\mathbf{u}_1, \mathbf{v}_1 \in \mathbb{R}^4$ and $\bar{\mathbf{u}}_2 = \mathbf{v}_2$. Moreover, $k_j \in \mathbb{R}$, $j = 1, \ldots, 3$, and $\bar{k}_4 = -\mathrm{i}k_2$. If $(a_1, a_2) \in \mathcal{R}_2$ then $\bar{\mathbf{u}}_j = \mathbf{v}_j$, $j = 1, 2$, and $k_1, k_2 \in \mathbb{R}$, $\bar{k}_3 = -\mathrm{i}k_1$, $\bar{k}_4 = s\mathrm{i}k_2$. Finally, if $(a_1, a_2) \in \mathcal{R}_3$, $\mathbf{u}_2 = \bar{\mathbf{u}}_1$, $\mathbf{v}_2 = \bar{\mathbf{v}}_1$ and $\bar{k}_2 = k_1$, $\bar{k}_4 = k_3$. The properties for k_j, $j = 1, \ldots, 4$, are given in Lemma 1. Using that, one can compute $\bar{S}_2$. □

7. Homographic solutions

In this section we consider homographic solutions of a planar three-body problem for some homogeneous potentials. After some reductions the nontrivial characteristic multipliers are given by a four-dimensional periodic linear system of the type (1) (see [7]). The normal form technique can be applied in order to get the boundaries of the stability regions. To do that we introduce briefly the homographic solutions to be studied (see [8,10]).

Let us consider the Hamiltonian system defined by the Hamiltonian function

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p} - U(\mathbf{q}),$$

where $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$, $\mathbf{q}_j, \mathbf{p}_j \in \mathbb{R}^2$, $j = 1, 2, 3$, denote the position and the conjugate momenta for the masses m_j, $M = \mathrm{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$ and

$$U(\mathbf{q}) = \sum_{1 \leqslant i < j \leqslant 3} \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|^\alpha},$$

with $0 < \alpha < 2$. It is not restrictive to assume that $m_1 + m_2 + m_3 = 1$.

Let us consider $\mathbf{q}_c$ a central configuration, that is, a solution of the equation $-M\mathbf{q} = \nabla U(\mathbf{q})$ (after suitable scalings). In a similar way to the Newtonian case, for any $\alpha \in (0, 2)$ there exist three collinear central configurations, with the masses on a straight line, and two triangular ones, where the masses are located at the vertices of an equilateral triangle. See [7] for some details. A solution of the Hamiltonian system is called homographic if the position of the masses at any time, $\mathbf{q}(t)$, is obtained from a central configuration, $\mathbf{q}_c$, by a rotation and an homothety. It is well known that they can be written as

$$\mathbf{q}(t) = r(t)\Omega\big(f(t)\big)\mathbf{q}_c, \quad \Omega = \mathrm{diag}(\Omega_1, \Omega_1, \Omega_1),\ \Omega_1 = \begin{pmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{pmatrix}, \tag{56}$$

where r is a solution of the potential equation

$$r'' = -\frac{dV}{dr}(r), \quad V(r) = -\frac{1}{\alpha r^\alpha} + \frac{\omega^2}{2r^2} \tag{57}$$

being $' = d/dt$, and $f(t) = \int_0^t \frac{\omega}{r(s)^2}\,ds$. We shall denote the energy of (57) by

$$E_K = \frac{(r')^2}{2} + V(r).$$

It is not restrictive to our purposes to consider $E_K = -1/2$. In the Newtonian case, that is, $\alpha = 1$, we get $r(f) = \omega^2/(1 + e\cos f)$, where e is the eccentricity and f is the true anomaly. Moreover, $\omega^2 = 1 - e^2$. So, the homographic solutions for $\alpha = 1$ can be parameterized by e.

In the general case, $0 < \alpha < 2$, once a central configuration has been fixed we get a family of homographic solutions that can be parameterized by $0 < \omega \leqslant \omega_c = ((2-\alpha)/\alpha)^{(2-\alpha)/2\alpha}$ (see [7]). We can introduce a generalized eccentricity

$$e := \sqrt{1 - \frac{\alpha}{2-\alpha}\omega^{2\alpha/(2-\alpha)}}.$$

We note that for $\omega = \omega_c$ one has a relative equilibrium solution. In this case, $e = 0$. Our results can be applied for $e \geqslant 0$ small enough.

Homographic solutions can be seen as equilibrium points by introducing a rotating and pulsating system. Using the integrals of the center of mass we can reduce to a nonautonomous linear system of order 8. In [7] it is proved that this system uncouples in 2 linear subsystems of order 4. We skip the details of this reduction and from now on we only consider the nontrivial subsystem, that is,

$$\dot{\mathbf{x}} = A(f)\mathbf{x}, \quad A(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(f) & -2J_2 \end{pmatrix},\ \tilde{A} = g^{\alpha-2}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{58}$$

where $\dot{} = d/df$, $g = \omega^{\frac{2}{2-\alpha}} r^{-1}$ and λ_1, λ_2 are constants which depend on the masses and the central configuration. Furthermore $g(f)$ is a periodic solution of

$$\ddot{z} = -\frac{d\mathcal{V}}{dz}(z), \quad \text{with } \mathcal{V}(z) = \frac{z^2}{2} - \frac{z^\alpha}{\alpha} \tag{59}$$

for the energy level $E = -(1/2)\omega^{2\alpha/(2-\alpha)}$ (see [7] for details). We note that $\mathcal{V}(z)$ has a minimum at $z = 1$ which corresponds to a relative equilibrium that is, $\omega = \omega_c$ and $e = 0$. Therefore, linear system (58) is of the form (1) with $t = f$ and $G_1 = G_2 = g^{\alpha-2}$. The small parameter ϵ will be here the (generalized) eccentricity e. We remark that for $e = 0$, $g(f)$ is constant and (58) is autonomous. For $e \neq 0$ small enough, in the Newtonian case we have $g(f) = 1 + e\cos f$ and d'Alembert property is trivially satisfied. In Section 7.2 we prove that this property is also satisfied in the general case, $0 < \alpha < 2$.

In both cases, triangular and collinear, λ_1, λ_2 depend on a mass parameter β_t and β_c, respectively, according to Table 2. We note that in the collinear case the mass parameter β_c depends on the solution of, the well-known Euler quintic's equation if $\alpha = 1$, and some generalization of this equation if $\alpha \neq 1$ (see [7] for details).

For a triangular configuration, (λ_1, λ_2) describes a segment with endpoints $(\alpha + 2, 0)$, $((\alpha+2)/2, (\alpha+2)/2)$, going from region $\mathcal{R}_2$ to $\mathcal{R}_3$, using the notation introduced in Section 1.1 (see Fig. 1). Table 3 summarizes the critical values of β_t such that bifurcations are expected for $e > 0$ small enough, in the nondegenerate cases. For the Newtonian case see Fig. 5 for a global description of the different kinds of linear stability. For other values of α we refer to [7].

In the collinear case, (λ_1, λ_2) describes a segment in the plane on the region $\mathcal{R}_1$ with endpoints $(\alpha+2, 0)$ and $((\alpha+1)2^{\alpha+2} + 1, 1 - 2^{\alpha+2})$ (see Fig. 1). Let us denote by ω the frequency. In this case, resonance can be attained when $\omega T = n\pi$ for some $n \in \mathbb{N}$. A simple computation shows that this is accomplished for $n \in \mathbb{N}$ satisfying

$$2 < n < \frac{2\omega_M}{\sqrt{2-\alpha}},$$

being $\omega_M = \sqrt{1 - 2^{\alpha+1}\alpha + 2^{\frac{\alpha}{2}}\sqrt{2^{\alpha+2}(\alpha+2)^2 - 8\alpha}}$ the maximum value of ω (see [7]). For these values of ω a transition EH $\leftrightarrow$ HH is expected. We note that in this case the number of resonant points increases as α increases tending to 2. The same occurs in the last case of Table 3 for a triangular configuration.

Table 2
Values of λ_1, λ_2 being $\kappa = m_1m_2 + m_1m_3 + m_2m_3$

Triangular	λ_1, λ_2 zeroes of $p(\lambda) = \lambda^2 - (\alpha+2)\lambda + \beta_t/4$, $\beta_t = 3(\alpha+2)^2\kappa$
Collinear	$\lambda_1 = (\alpha+1)\beta_c + \alpha + 2$, $\lambda_2 = -\beta_c$, $\beta_c \in (0, 2^{\alpha+2} - 1)$

Table 3
Resonances for $e = 0$ in the triangular case and expected transitions for small e

β_t^*	$\frac{3}{4}(2-\alpha)^2$	$(2-\alpha)^2$	$(2-\alpha)^2(n^2-1)^2$ for $n \in \mathbb{N}$ $2 \leqslant n \leqslant 2/\sqrt{2-\alpha}$
Transition	EE $\leftrightarrow$ EH	EE $\leftrightarrow$ CS	CS $\leftrightarrow$ HH

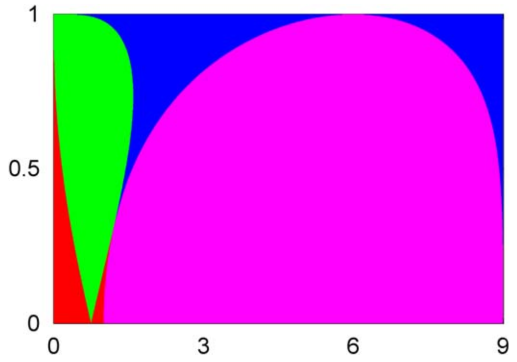

Fig. 5. Bifurcation diagram of the triangular Newtonian homographic solutions in the (β_t, e)-plane. The color code is the same used in Fig. 3.

7.1. The Newtonian case

Let us consider $\alpha = 1$. So, $g(f) = 1 + e\cos f$.

In the triangular case, we obtain a resonant point for $\beta_t^* = 3/4$. For this value, $(\lambda_1, \lambda_2) \in \mathcal{R}_2$ and the resonant frequency is $\omega_2 = 1/2$. Using the results of Section 4 we obtain that a resonant tongue of order $O(e)$ is born at the point $(\beta_t, e) = (3/4, 0)$, whose boundaries are given by

$$\beta_t^- = 3/4 - de + O(e^2), \qquad \beta_t^+ = 3/4 + de + O(e^2), \quad d = 1.313543288080\ldots. \tag{60}$$

Figure 5 shows the bifurcation diagram on the (β_t, e)-plane computed numerically. The behaviour for $e \lesssim 1$ is detailed in [7].

In the collinear case there are three resonant points corresponding to frequencies $\omega = 3/2, 2, 5/2$. First we consider the cases $\omega = 3/2$ and $\omega = 5/2$. Let β_c^* the value of β_c at resonance. By taking $\beta_c = \beta_c^* + \delta$, from (25) we get the following boundaries of the resonant tongues

$$\beta_c - \beta_c^* = -0.4208699\ldots e^2 \pm 0.0336193\ldots e^3 + O(e^5) \quad \text{if } \omega = 3/2,$$
$$\beta_c - \beta_c^* = -1.9578204\ldots e^2 - 0.5109419\ldots e^4 \pm 0.0003288\ldots e^5 + O(e^6) \quad \text{if } \omega = 5/2.$$

We note that, in agreement with the results of Section 4, the resonant tongues $\mathcal{T}_{3/2}$, $\mathcal{T}_{5/2}$ are of order $O(e^3)$, $O(e^5)$, respectively, due to the fact that the suitable coefficient is different from zero.

The existence of the tongue (60) in the triangular case was proved by Roberts [9] in a different way. We note that in this case only lower order terms in e are needed. However, to detect $\mathcal{T}_{3/2}$ and $\mathcal{T}_{5/2}$ in the collinear case, one has to compute terms of order 3 and 5, respectively, in the eccentricity. In these cases, the method used in [9] becomes impracticable.

Now we study the case $\omega = 2$ for the collinear solutions. Although λ_1, λ_2 depend on the mass parameter β_c or β_t, we can consider the system for arbitrary $(\lambda_1, \lambda_2) \in \mathcal{R}_1 \cup \mathcal{R}_2$. To prove Theorem 3 we shall use the following lemma.

Lemma 8. *Assume that* (58) *has a 2π-periodic solution, φ, for a fixed value of $e \in (0, 1)$ and $\lambda_j \neq 0$, $j = 1, 2$. Then, there exists a second periodic solution with the same period which is independent of φ.*

Proof. System (58) can be written as the following system of second order equations:

$$(1+e\cos f)\ddot{x}_1 = \lambda_1 x_1 - 2\dot{x}_2(1+e\cos f),$$
$$(1+e\cos f)\ddot{x}_2 = \lambda_2 x_2 + 2\dot{x}_1(1+e\cos f). \tag{61}$$

A 2π-periodic solution of the system above can be written as

$$x_1(f) = a_0 + \sum_{n\geqslant 1} a_n \cos(nf) + \sum_{n\geqslant 1} b_n \sin(nf),$$
$$x_2(f) = c_0 + \sum_{n\geqslant 1} c_n \cos(nf) + \sum_{n\geqslant 1} d_n \sin(nf). \tag{62}$$

Then, the coefficients must satisfy the following uncoupled sets of recurrences:

$$\begin{aligned} &\lambda_1 a_0 = e\left(d_1 - \frac{a_1}{2}\right),\\ &eA_2\mathbf{u}_2 = B_1\mathbf{u}_1,\\ &eA_{n+1}\mathbf{u}_{n+1} = B_n\mathbf{u}_n - eA_{n-1}\mathbf{u}_{n-1}, \quad n\geqslant 2,\ \mathbf{u} = (a_n, d_n)^T, \end{aligned} \tag{63}$$

$$\begin{aligned} &\lambda_2 c_0 = -e\left(b_1 + \frac{c_1}{2}\right),\\ &eA_2S\mathbf{v}_2 = B_1S\mathbf{v}_1,\\ &eA_{n+1}S\mathbf{v}_{n+1} = B_nS\mathbf{v}_n - eA_{n-1}S\mathbf{v}_{n-1}, \quad n\geqslant 2,\ \mathbf{v} = (b_n, c_n)^T, \end{aligned} \tag{64}$$

where $A_n = -\frac{n}{2}\left(\begin{smallmatrix} n & -2\\ -2 & n\end{smallmatrix}\right)$, $B_n = \left(\begin{smallmatrix} \lambda_1+n^2 & -2n\\ -2n & \lambda_2+n^2\end{smallmatrix}\right)$ and $S = \mathrm{diag}(1,-1)$.

We note that if $\mathbf{u}_n$, $n\geqslant 1$, is a nontrivial solution of the last two equations in (63) then $\mathbf{v}_n = S\mathbf{u}_n = (a_n, -d_n)^T$, $n\geqslant 1$, is a nontrivial solution of the second and third equations in (64). Moreover, A_n is a nonsingular matrix for $n>2$. However, $\det(A_2)=0$. But if $\det(B_1) = (\lambda_1+1)(\lambda_2+1)-4\neq 0$, given $\mathbf{u}_2$ we can compute $\mathbf{u}_1$ from the second equality in (63), and from the last equation we obtain $\mathbf{u}_n$ for $n\geqslant 3$.

We assume that (62) is a nontrivial 2π-periodic solution of (61). Then, both (63) and (64) have a solution. We assume that (63) admits a nontrivial solution. Then, $\sum_{n\geqslant 1} a_n\cos(nf)$ and $\sum_{n\geqslant 1} d_n \sin(nf)$ are convergent. Therefore $\mathbf{v}_n = S\mathbf{u}_n$, that is, $b_n = a_n$ and $c_n = -d_n$, for $n\geqslant 1$, is a solution of (64). Then, we can built two independent periodic solutions of (61) as

$$x_1^{(1)}(f) = a_0 + \sum_{n\geqslant 1} a_n\cos(nf), \qquad x_2^{(1)}(f) = \sum_{n\geqslant 1} d_n \sin(nf),$$
$$x_1^{(2)}(f) = \sum_{n\geqslant 1} a_n\sin(nf), \qquad x_2^{(2)}(f) = c_0 - \sum_{n\geqslant 1} d_n\cos(nf), \tag{65}$$

where $a_0 = \frac{e}{\lambda_1}(d_1 - \frac{a_1}{2})$ and $c_0 = \frac{e}{\lambda_2}(\frac{d_1}{2} - a_1)$. $\square$

Proof of Theorem 3. For $\omega = n$, $n \in \mathbb{N}$, one stability parameter, tr_2, is equal to 2 for $e = 0$. Then the boundaries of the resonant region are defined by $\mathrm{tr}_2 = 2$. Furthermore, if $(\lambda_1, \lambda_2, e)$ belongs to the boundary, the linear system (58) has a 2π-periodic solution.

Let us define $\Phi(2\pi)$ the monodromy matrix of (58). After Lemma 8, if $(\lambda_1, \lambda_2, e)$ belongs to the boundary of the resonant region then $\Phi(2\pi)$ can be written (in a suitable basis) as

$$\Phi(2\pi) = \begin{pmatrix} Q & 0 \\ 0 & I_2 \end{pmatrix},$$

for some 2×2 matrix Q. Using the normal form we can compute $\Phi(2\pi)$ up to a given order in δ_1, δ_2, e. As we are in a single resonance case we know that the reduced system becomes uncoupled. Assume that $(a_1, a_2) \in \mathcal{R}_1$. Then the subsystem that defines tr_2 is (24) (in the case $(a_1, a_2) \in \mathcal{R}_2$ a similar subsystem is obtained). We define for this system the symplectic change of coordinates

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then the new system is

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = S_1 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

where $S_1 = \begin{pmatrix} 0 & \sigma_2 - 2\sigma_3 \\ -(\sigma_2 + 2\sigma_3) & 0 \end{pmatrix}$. The corresponding monodromy matrix is $\exp(2\pi S_1)$.

Let us assume that $(\lambda_1, \lambda_2, e)$ belongs to the boundary such that $\sigma_2 - 2\sigma_3 = 0$. Then, $S_1 = \begin{pmatrix} 0 & 0 \\ -(\sigma_2 + 2\sigma_3) & 0 \end{pmatrix}$ and $\exp(2\pi S_1) = \begin{pmatrix} 1 & 0 \\ -2\pi(\sigma_2 + 2\sigma_3) & 1 \end{pmatrix}$.

Assume that for these values of the parameters, $\sigma_2 + 2\sigma_3 \neq 0$. Then system (58) would have a unique 2π-periodic solution. This gives a contradiction with Lemma 8. In this way we have proved that the two boundaries coincide up to an arbitrary order in e, once $\delta_1 = \delta_1(e)$ and $\delta_2 = \delta_2(e)$. Using the analyticity they coincide for any value of the eccentricity. □

The left part of Fig. 6 shows the bifurcation diagram on the (β_c, e)-plane computed numerically for $\beta_c \in (0, 7)$, $e \in [0, 1)$. The first tongue is born at $\beta_c^* = 3(\sqrt{41} - 1)/16 = 1.013\ldots$, which corresponds to $\omega = 3/2$. We recall that the width of $\mathcal{T}_{3/2}$ is of order e^3. So, to distin-

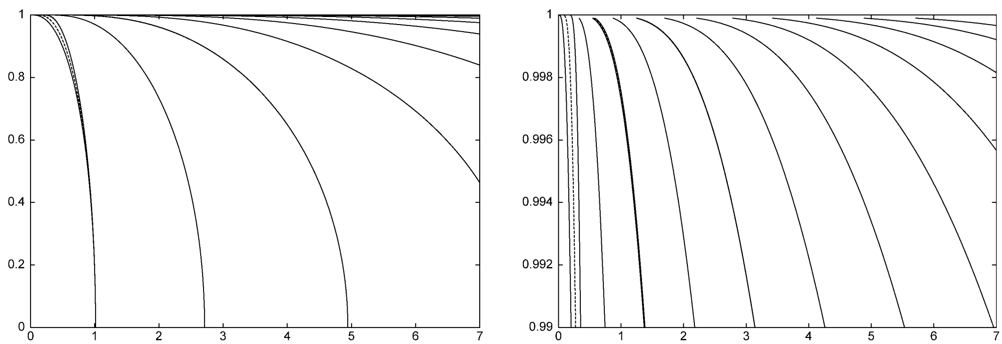

Fig. 6. Left: resonant tongues in the (β_c, e)-plane for the collinear Newtonian homographic solutions. Right: a magnification for e close to 1.

guish the two boundaries we have to look at big values of the eccentricity. In the figure the line inside the resonant tongue corresponds to a minimum of the stability parameter. The second 'tongue' $\mathcal{T}_2$, is only a curve defined by points (β_c, e) for which the second stability parameter is equal to 2, as predicted by Theorem 3. For the third tongue $\mathcal{T}_{5/2}$ the width is of order e^5. We can distinguish the two boundaries in the magnification displayed on the right part of Fig. 6 for big values of e. Other curves in these plots are resonant tongues $\mathcal{T}_\omega$ for $\omega = \frac{m}{2}$, $m \in \mathbb{N}$, $m > 5$. They are born at values $\beta_c^* > 7$ and, hence, they are not relevant for small values of e. However, infinitely many resonant zones enter the domain when e increases. The behaviour of $\mathcal{T}_\omega$ as e goes to 1 is described in [7].

7.2. *The general case*

For the general case we do not know explicitly the expression of $g^{\alpha-2}$. In this section we shall see that $g^{\alpha-2}$ satisfies d'Alembert property, and then we can use the results given in Section 4 to compute the boundaries of the resonant regions.

Let $g(f)$ be the solution of (59) such that $\dot{g}(0) = 0$ and $g(0)$ is the minimum of $g(f)$. We introduce a new variable $v = g^{\alpha-2} - 1$. Then, the second order equation for v is

$$\ddot{v} = 2(\alpha-2)(\alpha-3)E(v+1)^{\frac{4-\alpha}{2-\alpha}} + (\alpha-2)^2(v+1)\left(\frac{3}{\alpha}(v+1) - 1\right), \tag{66}$$

where E denotes the energy of (59), that is, $E = \frac{\dot{z}^2}{2} + \mathcal{V}(z)$.

Let $\varepsilon > 0$ be small enough. We look for a solution of (66) which satisfies initial conditions $v(0) = \varepsilon$ and $\dot{v}(0) = 0$. We shall write

$$v(f) = v_1(f)\varepsilon + v_2(f)\varepsilon^2 + v_3(f)\varepsilon^3 + \cdots, \tag{67}$$

where $v_1(0) = 1$, $v_j(0) = 0$ for $j \geqslant 2$ and $\dot{v}_j(0) = 0$ for $j \geqslant 1$. We remark that writing the energy of (59) in terms of v we have that

$$E = \frac{1}{2}(\varepsilon+1)^{\frac{2}{\alpha-2}} - \frac{1}{\alpha}(\varepsilon+1)^{\frac{\alpha}{\alpha-2}} = E_1 + \Delta, \quad E_1 = -\frac{2-\alpha}{2\alpha}, \tag{68}$$

and $\Delta = \alpha_2\varepsilon^2 + \alpha_3\varepsilon^3 + \alpha_4\varepsilon^2 + O(\varepsilon^5)$ with

$$\alpha_2 = \frac{1}{2(2-\alpha)}, \qquad \alpha_3 = -\frac{4-\alpha}{3(2-\alpha)^2}, \qquad \alpha_4 = \frac{(4-\alpha)(3-\alpha)}{4(2-\alpha)^2}, \quad \ldots$$

To get $v(f)$ we use a Lindstedt–Poincaré method. So, we introduce a new independent variable $\tau = \nu f$ with

$$\nu = \nu_0 + \nu_1\varepsilon + \nu_2\varepsilon^2 + \cdots.$$

The coefficients ν_j, $j \geqslant 0$ will be determined in order to eliminate resonant terms. Using (68) Eq. (66) can be written as

$$\nu^2\frac{d^2v}{d\tau^2} = f(v) + g(v)\Delta, \tag{69}$$

where

$$f(v) = E_1 g(v) + (\alpha - 2)^2 (v+1)\left(\frac{3}{\alpha}(v+1) - 1\right),$$
$$g(v) = 2(2-\alpha)(3-\alpha)(v+1)^{\frac{4-\alpha}{2-\alpha}}.$$

By substituting (67) in (69) we get

$$\nu_0^2 \frac{d^2 v_1}{d\tau^2} = -(2-\alpha) v_1, \qquad v_1(0) = 1, \quad \frac{dv_1}{d\tau}(0) = 0.$$

We choose $\nu_0^2 = (2-\alpha)$ and then trivially $v_1(\tau) = \cos\tau$. In a similar way we get

$$\begin{aligned} v_2(\tau) &= \frac{1}{2(2-\alpha)} + \frac{\alpha-4}{3(2-\alpha)}\cos\tau - \frac{2\alpha-5}{6(2-\alpha)}\cos(2\tau), \\ v_3(\tau) &= \frac{\alpha-4}{3(\alpha-2)^2} + \left(\frac{(\alpha-4)(7-\alpha)}{9(2-\alpha)^2} - \frac{9\alpha^2-47\alpha+62}{96(2-\alpha)^2}\right)\cos\tau \\ &\quad - \frac{(2\alpha-5)(\alpha-4)}{9(2-\alpha)^2}\cos(2\tau) + \frac{9\alpha^2-47\alpha+62}{96(2-\alpha)^2}\cos(3\tau), \end{aligned}$$

$\nu_1 = 0$ and

$$\nu_2 = -\frac{\sqrt{2-\alpha}}{2(2-\alpha)^2}\left(\frac{1}{6}(2\alpha-5)(11-2\alpha) - \frac{3}{4}(\alpha-3)(4-\alpha)\right).$$

In this way we can obtain $g^{2-\alpha} = 1 + v(\tau)$ up to a given order. Then, $g^{2-\alpha} = 1 + v(\nu f)$ is a periodic function of f with period $T = \frac{2\pi}{\nu}$.

Now we shall see that $g^{2-\alpha}$ is an even function of f and satisfies the d'Alembert property.

Lemma 9. *Let $v(\tau) = \sum_{m\geqslant 1} v_m(\tau)\varepsilon^m$ be the solution of* (69) *such that $v_1(0) = 1$, $v_j(0) = 0$ for $j \geqslant 2$ and $\dot v_j(0) = 0$ for $j \geqslant 1$. Then, $v_m(\tau)$, $m \in \mathbb{N}$, is an even function on τ which satisfies the d'Alembert condition, that is, for $m \in \mathbb{N}$,*

$$v_m(\tau) = \sum_{l=0}^{m} a_{ml}\cos(l\tau). \tag{70}$$

Proof. We know that $g(f)$ is an even periodic function of f. So, $v(\tau)$ is also an even function. Moreover $v_1(\tau) = \cos\tau$. Assume that $v_m(\tau)$ for $m = 1, 2, \dots, k-1$ are known and satisfy (70). If we define $w = e^{i\tau}$ then $v_m(\tau)$ contains terms w^l with $l \leqslant m$.

The equation for $v_k(\tau)$ is obtained by equating in (69) terms of order k in ε. It is clear that $v_1(\tau), \dots, v_{k-1}(\tau)$ give terms with w^l, with $l \leqslant k-1$, in $\ddot v$.

Concerning the right part of (69) to get the terms of order k in ε from $f(v)$ it is sufficient to consider

$$f(v)=f'(0)v_k(\tau)+\sum_{j=2}^{k}\frac{f^{(j)}(0)}{j!}\big(v^{(k)}\big)^j,$$

where $v^{(k)}(\tau)=v_1(\tau)\varepsilon+\cdots+v_k(\tau)\varepsilon^k$.

The terms of order k in ε which come from $(v^{(k)})^j$ can be written as

$$\big(v^{(k)}\big)^j=\sum_{\substack{l_1+\cdots+l_k=j\\ l_1+2l_2+\cdots+kl_k=k}} v_1^{l_1}v_1^{l_2}\cdots v_k^{l_k}\varepsilon^k. \tag{71}$$

In (71) we consider $j\geqslant 2$. This implies $l_k=0$ in the sum (71). Using the hypothesis on $v_1(\tau),\dots,v_{k-1}(\tau)$ we get that the highest term in w which appears in $v_1^{l_1}v_2^{l_2}\cdots v_k^{l_k}$ is $w^{l_1+2l_2+\cdots+(k-1)l_{k-1}}=w^k$. In a similar way it can be proved that $g(v)\Delta$ contributes to the equation of v_k with terms w^l, $l\leqslant k-2$. Therefore we can write the equation for $v_k(\tau)$ as a linear nonhomogeneous differential equation

$$\nu_0^2\ddot{v}_k=f'(0)v_k+F(\tau),$$

where $F(\tau)$ depends on $v_1(\tau),\dots,v_{k-1}(\tau)$. The terms of $F(\tau)$ contain w^l with $l\leqslant k$. This proves the lemma. □

Remark 11. In contrast with the Newtonian case, if $\alpha\neq 1$ the second periodic solution given by Lemma 8 does not exist. Analysis of the corresponding normal forms shows that the tongues are open.

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